

# Consumption I: Background, History and Stochastic Theory

*"Consumption is the sole end and purpose of all production."*

Adam Smith

You already know a lot about optimal consumption theory from lectures on Dynamic General Macroeconomic Equilibrium, Neoclassical Growth Theory with Endogenous Saving, and Overlapping Generations. The value added to be supplied here is (i) a brief exposition of the most important historical contributions and, more important, (ii) an emphasis on the modern stochastic view in which expectations about future income play a central role. (Taking incomes as given is obviously a highly unrealistic assumption.) We shall also use a new solution approach (Bellman's value function).

## 1 Background Stylized Facts

- Consumption comprises around 2/3 of GDP. For this reason alone it is worthy of your attention. Consumption is of course the flip side of saving,  $S = (Q - C)$ , and for the same reason saving is accurately thought of as *future* consumption. So when you model one, you are also modeling the other displaced in time. And as you already know from the growth theory lectures, saving is the driving force of capital formation and, therefore, ultimately of living standards.
- Consumption expenditure on non-durable goods (the variable in national accounts and budget studies that is closest to the modern concept of "consumption services") is significantly less volatile than disposable income.
- Consumption of nondurables varies a lot less than Consumption of durables (which is about 12-15% of total consumption in mature economies) because the later has an important investment (in future consumption services) component and, consequently, it is more discretionary. Durables consumption itself is much more volatile than disposable income.

For example, in some US and UK data (the US and UK are probably the most thoroughly studied economies) the standard deviations of annual growth rates of *aggregate* disposable income and consumption expenditures over 1960-96 are:<sup>1</sup>

| Standard Deviations of:                               | US    | UK    |
|---|-------|-------|
| $\sigma [\Delta \ln QD]$ ( <i>disposable income</i> ) | 0.025 | 0.026 |
| $\sigma [\Delta \ln C\_NonDurables\ Consumption]$     | 0.018 | 0.021 |
| $\sigma [\Delta \ln C\_Durables\ Consumption]$        | 0.069 | 0.112 |

A similar pattern is exhibited by micro data from budget studies undertaken from the late 1960s to early 1990s. After netting out birth-cohort and age effects (by polynomial regression), the standard deviations of changes in log income and log consumption are:<sup>2</sup>

| Standard Deviations of:   | US   | UK   |
|---|------|------|
| $\sigma [\Delta \ln qd]$ ( <i>disposable income</i> )                     | 3.68 | 3.60 |
| $\sigma [\Delta \ln c]$ ( <i>total consumption per adult equivalent</i> ) | 2.39 | 2.64 |
| $\sigma [\Delta \ln c\_NonDurables\ Consumption\ per\ adult\ equivalent]$ | 1.95 | 2.05 |
| $\sigma [\Delta \ln c\_Durables\ per\ adult\ equivalent]$                 | 15.8 | 8.54 |

- Consumption is "flatter" (less hump-shaped) than income over the life cycle (with consumption and income both peaking at the mid-40s of age), yet it exhibits 'excess sensitivity' to current income by standards of a strict version of the 'permanent income hypothesis'.<sup>3</sup>

## 2 History: John Maynard Keynes

J.M. Keynes' model of consumption<sup>4</sup> is

$$c_{it} = \alpha + \beta \cdot qd_{it}, \quad \alpha > 0, \quad 0 < \beta < 1. \quad (1)$$

<sup>1</sup>The source of the two data tables is Attanasio in Taylor and Woodford, eds., 1999, Chapter 11, Tables 1 and 5.

<sup>2</sup>In multi-person households 'per adult equivalent' consumption is calibrated by applying the weights 1.0. (1st adult), 0.69 (2nd adult) and 0.43 (children under 19). Henceforth I'll use lower case  $c$  and  $qd$  when the argument pertains to the consumption and disposable income of individual agents.

<sup>3</sup>See, for example, the US and UK data reported in the Attanasio chapter cited in the last note.

<sup>4</sup>Oddly, Keynes was somewhat hostile to mathematical and quantitative economics, despite having been well educated in mathematics and probability at Eton and Cambridge (though not

Recall that in the Keynesian system the proposition that the marginal propensity to consume is less than 1.0 ( $0 < \beta < 1$ ) originates with IS analysis (the  $i, Q$  relation in the goods market, such that at equilibrium planned expenditure, that is, demand or desired purchases,  $E$ , equals output produced,  $Q$ ). In other words, it originates with the "Keynesian cross":<sup>5</sup>

$$Q \equiv C + I + G \quad (2)$$

$$E = E[(Q - T), I(i - p^e), G], \quad 0 < E'(Q - T) < 1. \quad (3)$$

Remember also that under Keynes the Average Propensity to Consume (APC) falls with income; the rich save more than the poor:

$$APC_{it} \equiv \frac{c_{it}}{qd_{it}} = \frac{\alpha}{qd_{it}} + \beta \quad (4)$$

$$\frac{\partial APC_{it}}{\partial qd_{it}} = -\frac{\alpha}{qd_{it}^2} < 0. \quad (5)$$

This predictions of the 'Keynesian model' work out pretty well in cross-section data on individuals (across  $i$ ); a regression based on the model

$$c_i|t = \alpha|t + \beta|t \cdot qd_i|t, \quad (6)$$

yields estimates  $\alpha_i > 0$ ,  $0 < \beta < 1$ .

However, corresponding estimates obtained from an aggregate (macroeconomic) time series regression model,

$$C_t = \alpha + \beta \cdot QD_t, \quad (7)$$

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in probability theory as we think of that subject today). You will find surprisingly few equations his magnum opus *The General Theory of Employment, Interest and Money* (1936). Mathematical statements of Keynesian thinking came at the hands of others; initially the IS-LM representation of Sir John Hicks in (*Econometrica*, 1937). Hicks himself expressed grave misgivings about formalized economics toward the end of his career). Moreover, Keynes never took an economics degree and, in fact, had little formal education at all in economics – just one 8 week course in 1905 with Alfred Marshall at Cambridge. (Then again, Marshall was perhaps the greatest economist of his age, and so he was a pretty good guy to get a one course education from.) Arthur Cecil Pigou succeeded Marshall to the Cambridge Professorship of 'Political Economy' (in those days there was only one chair in most fields), and Keynes never seriously entertained leaving King's College, never mind the University of Cambridge. So he was never a Professor either (just a Fellow of King's). Keynes learned his economics on the job; he was an autodidactic genius who almost single-handedly invented what we now call 'macroeconomics'. Bow your head a little when his name comes to mind.

<sup>5</sup>Below I neglect exports net of imports (closed economy).

tend to yield results showing  $1 < \hat{\beta}(t) \gg \hat{\beta}(i)|t$ ,  $0 \leq \hat{\alpha}(t) \ll \hat{\alpha}(i)|t$ .<sup>6</sup> [*Graph illustrating the cross-section/time-series inconsistency*] One should be thankful for this outcome, otherwise the macroeconomy would surely have stagnated with the secular rise in incomes. If the APC did fall with income as the Keynesian model implies, saving would have come to comprise a larger and larger fraction of income as the economy grew, and it most likely would not have been absorbed by sufficient opportunities for profitable investment. The mountains of idle saving would drive macroeconomy into stagnation, and it eventually would collapse in prolonged depression. (Simon Kuznets' "secular stagnation".)

## 2.1 Short Run Fiscal Policy Implications of the Keynesian Consumption Function

Since  $c_t$  is a linear function of  $qd_t$ , anything that changes current disposable income will change current consumption. So a government that wanted, say, to boost aggregate demand during a cyclical contraction could exert great leverage on the macroeconomy by cutting taxes or increasing transfers to households, whose time  $t$  consumption would rise by  $\beta \cdot \Delta QD_t$ . As we shall see ahead, effects of fiscal activism are weaker under Friedman's permanent income hypothesis, and matters get even bleaker under subsequent views of consumption, particularly when consumption behavior is joined to the 'Ricardian equivalence' proposition. Stay tuned.

## 3 History: Milton Friedman's Permanent Income Hypothesis<sup>7</sup>

The publication of Friedman's *A Theory of the Consumption Function* in 1957 was a huge breakthrough, helping earn him the Nobel Memorial Prize in economics a half generation later.<sup>8</sup> Among many other things, Friedman's consumption model – the famous "permanent income hypothesis" (PIH) – helped resolve the paradox of

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<sup>6</sup>Recall the least-squares formula for the intercept estimate:  $\hat{\alpha} = \bar{C} - \hat{\beta} \bar{QD}$ . Hence  $\hat{\beta}(t) \gg \hat{\beta}(i)|t$  implies the reverse inequality for  $\hat{\alpha}$ , and as  $\hat{\beta} \rightarrow 1$  the APC no longer tends to fall with income.

<sup>7</sup>The PIH is closely related to Modiglianni and Brumberg's (1954) s life-cycle model. In fact we usually speak of the life cycle-permanent income hypothesis.

<sup>8</sup>Friedman and Schwartz's monumental *A Monetary History of the United States* (1963), which documented the potency of monetary policy in explaining the American business cycle, was also acknowledged by the Nobel Committee.

different time-series and cross-section 'consumption functions'. The two equation system can be written<sup>9</sup>

$$c_{it} = \alpha + \beta \cdot qd_{it}^P \quad (8)$$

$$qd_{it} = qd_{it}^P + qd_{it}^T \quad (9)$$

where  $qd^P$  denotes "permanent" income,  $qd^T$  denotes "transitory" income (random shocks to permanent income that are uncorrelated with  $qd^P$ ). The PIH predicts  $\alpha = 0$ ,  $\beta = 1$ . In other words, agents just consume their permanent income each period:<sup>10</sup>

$$c_{it} = qd_{it}^P \quad (10)$$

How did Friedman's theory resolve the paradox? The answer hinges on a classic 'errors in variables' problem that you surely have seen (or should have seen) in econometrics. Assume (8)-(9) is the true model, but you estimate by OLS the 'Keynesian' equation

$$c_{it} = \alpha + \beta \cdot qd_{it}.$$

OLS delivers the MPC estimate

$$\begin{aligned} \hat{\beta}_{OLS} &= \frac{Cov[c_{it}, qd_{it}]}{Var(qd_{it})} \quad (11) \\ &= \frac{Cov[qd_{it}^P, (qd_{it}^P + qd_{it}^T)]}{[Var(qd_{it}^P) + Var(qd_{it}^T) + 2Cov(qd_{it}^P, qd_{it}^T)]} \\ &= \frac{Var(qd_{it}^P)}{[Var(qd_{it}^P) + Var(qd_{it}^T)]} = \frac{1}{1 + Var(qd_{it}^T)/Var(qd_{it}^P)} < 1. \end{aligned}$$

So the MPC estimate is biased downward in proportion to the reciprocal of the signal-to-noise ratio. By contrast, under the PIH,  $c_{it} = qd_{it}^P$ , and assuming we have a proper measure of permanent income,  $qd_{it}^P$ , OLS of course yields

$$\begin{aligned} \hat{\beta}_{OLS} &= \frac{Cov[c_{it}, qd_{it}^P]}{Var(qd_{it}^P)} \quad (12) \\ &= \frac{Cov[qd_{it}^P, qd_{it}^P]}{Var(qd_{it}^P)} = \frac{Var(qd_{it}^P)}{Var(qd_{it}^P)} = 1. \end{aligned}$$

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<sup>9</sup>One could also partition observed consumption into permanent and transitory components. But not much insight is gained by adding this complication.

<sup>10</sup>Actually Friedman (and others who followed) define consumption as use of consumption services, not purchase of consumption goods, as in IS analysis. The distinction can be very important in empirical work (as the stylized facts shown earlier imply), and it creates a wedge between the Keynesian system and subsequent views.

In the cross-section (a regression over households  $i$  at some time  $t$ ) one would expect the variance of transitory income to be larger than in an aggregate time-series (over  $t$  with data based on national accounts aggregates or averages taken over  $i$ ) because random-transitory shocks to incomes would tend to cancel out across individuals and over business cycles. In aggregate time series the variance of transitory income in relation to the variance of permanent income will consequently be much smaller than in cross-sections of individual households, and so estimates  $\hat{\beta}$  will be closer to the "true" MPC of 1.0. [*Examples*] Hence, Friedman's PIH was widely viewed as reconciling the conflict between cross-section and time series estimates of consumption functions.

Friedman proposed (as "esthetic") an adaptive or partial adjustment view of how agents calibrate their permanent income. Assuming perceptions of  $qd_{it}^P$  are formed before realizations of  $qd_{it}$ , we have:

$$qd_{it}^P - qd_{it-1}^P = (1 - \lambda) \cdot (qd_{it-1} - qd_{it-1}^P) \quad (13)$$

$$qd_{it}^P = \lambda qd_{it-1}^P + (1 - \lambda) qd_{it-1} \quad (14)$$

$$\begin{aligned} qd_{it}^P &= \frac{(1 - \lambda)}{(1 - \lambda L)} qd_{it-1} \\ &= (1 - \lambda) \cdot \sum_{j=0}^{\infty} \lambda^j qd_{it-1-j}. \end{aligned} \quad (15)$$

### 3.1 Muth's Demonstration

Remember that Friedman wrote before the appearance of rational expectations in economics. In a famous article, John Muth (1960, JASA), who is regarded by many as the parent of rational expectations,<sup>11</sup> pointed out that an adaptive model of permanent income is an optimal predictor and consistent with the fundamental property of rational expectations, namely,

$$qd_{it}^P = qd_{it}^P | Adaptive + \varepsilon_{it}, \quad \varepsilon_{it} \sim 0, white \quad (16)$$

only if income obeyed an ARIMA(0,1,1) process:<sup>12</sup>

$$qd_{it} (1 - L) = \varepsilon_{it} (1 - \lambda L), \quad \varepsilon_{it} \sim 0, white. \quad (17)$$

<sup>11</sup>Especially, Muth *Econometrica*, 1961.

<sup>12</sup>Note a constant ("drift rate") could have been added to adaptive scheme and the ARIMA(0,1,1) model to accommodate secular rise in incomes due to economic growth, and the results below would pass through. Alternatively, one could view the data as having been purged of trend (and seasonal factors).

To see why, note that eq.(17) implies (showing all the Mickey Mouse steps)

$$\frac{(qd_{it} - qd_{it-1})}{(1 - \lambda L)} = \varepsilon_{it} \quad (18)$$

$$\frac{qd_{it}}{(1 - \lambda L)} = \frac{qd_{it-1}}{(1 - \lambda L)} + \varepsilon_{it} \quad (19)$$

$$\sum_{j=0}^{\infty} \lambda^j qd_{it-j} = \sum_{j=0}^{\infty} \lambda^j qd_{it-1-j} + \varepsilon_{it} \quad (20)$$

$$qd_{it} + \lambda \cdot \sum_{j=0}^{\infty} \lambda^j qd_{it-j-1} = \sum_{j=0}^{\infty} \lambda^j qd_{it-1-j} + \varepsilon_{it} \quad (21)$$

$$qd_{it} = (1 - \lambda) \sum_{j=0}^{\infty} \lambda^j qd_{it-1-j} + \varepsilon_{it}. \quad (22)$$

Eq.(22) indeed establishes Muth's claim because, as noted before, Friedman's proposed calibration of permanent income,

$$qd_{it}^P = (1 - \lambda) \cdot \sum_{j=0}^{\infty} \lambda^j qd_{t-1-j},$$

and the rational expectation (the expectation conditioned on information available up to and including period  $(t - 1)$ ) of time  $t$  income from the ARIMA(0,1,1) income process in eq.(17) is

$$\begin{aligned} E_{t-1} [qd_{it}] &= E_{t-1} \left[ (1 - \lambda) \sum_{j=0}^{\infty} \lambda^j qd_{it-1-j} + \varepsilon_{it} \right] \\ &= (1 - \lambda) \sum_{j=0}^{\infty} \lambda^j qd_{it-1-j} = qd_{it}^P | \text{Friedman}. \end{aligned} \quad (23)$$

### 3.2 Short-Run Fiscal Policy Implications of Friedman's PIH with Adaptive Income Expectations

Since  $c_t$  is now determined by  $qd_t^P$ , not  $qd_t$ , fiscal policy can change consumption (and thereby affect aggregate demand) only by changing agents' calibrations of their permanent disposable income. So a government that wants to revive aggregate demand during a contraction will have less leverage on the macroeconomy by

cutting taxes or increasing transfers to households, whose time  $t$  consumption (if permanent income is calibrated adaptively) will rise by  $\beta \cdot (1 - \lambda) \cdot \Delta qd_t$  (or by  $(1 - \lambda) \cdot \Delta qd_t$  at  $\beta = 1$ ), not by  $\beta \cdot \Delta qd_t$  as in the Keynesian consumption function. So Friedman's PIH – here with adaptive calibration of permanent income – implies a smaller contemporaneous response of consumption to a policy-induced shift in  $qd$  than the Keynesian consumption function. Nonetheless, since we do not yet have a formal intertemporal budget constraint, even predictable fiscal reactions to macroeconomic events can still move consumption, and since consumption is 2/3 or more of total income, still move the macroeconomy. Twenty minutes from now, after I have laid out the essentials of a subsequent modern stochastic view of consumption, fiscal authorities will be deprived of this capacity.

## 4 Stochastic Theory

Modern theory views consumption as intertemporal allocation problem.<sup>13</sup> In most respects it dates from Robert E. Hall's seminal 1978 JPE paper, which probably forever changed how the topic will be studied. Let me begin by showing what is necessary to obtain a result essentially identical to Friedman's main idea under the new regime initiated by Hall. (Henceforth I drop the subscript  $i$ ). In order to obtain  $c_t = qd_t^P$  we will need:

- Time-separability of utility of consumption, in other words, period (instantaneous) utility at each time  $t$  depends only on consumption in period  $t$ , it is not affected by consumption in other periods. (This rules out, for example, habit persistence and other forms of reference-dependent utility of consumption.)<sup>14</sup>
- Linearity of  $u' [c]$ , which means that the third derivative  $u'''(c)$  is zero. This implies that consumers do not respond to the amount of income risk they face – a property known as 'certainty equivalence'.
- $r = \rho$ , the real interest rate is constant and equal to the rate of time preference.
- Consumers face no liquidity constraints; they are free to lend and borrow at rate  $r$  throughout their lives (subject to a solvency constraint).

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<sup>13</sup>Setting up economic choice problems intertemporally originates before WW II in the work of the great Yale University economist Irving Fisher. (Fisher was a lousy investor though, and Yale wound up buying his house back from the bank for him after he lost everything following the 1929 stock market crash.)

<sup>14</sup>You will get a lecture on reference-dependent utility of consumption a little later on.

If the state of the world has the above characteristics, we will obtain

$$c_t = qd_t^P = \frac{r}{(1+r)} \cdot \left[ E_t \sum_{j=0}^{\infty} \frac{qd_{t+j}}{(1+r)^j} + a_t \right] \quad (24)$$

where  $a_t$  is current wealth, and the expectations operator  $E_t$  denotes expectations conditioned on all information available to the agent at time  $t$ , which in this lecture we shall assume includes knowledge of the realizations of all variables dated at time  $t$  or earlier. Note that the right-side of (24) is the annuity value of wealth (the permanently sustainable level of wealth given fixed  $r$ ).

## 4.1 A Standard Setup

Agents seek to maximize the present discounted value of expected utility of consumption<sup>15</sup>

$$\underset{\{c_{t+j}\}_{j=0}^{\infty}}{\text{Max}} E_t \sum_{j=0}^{\infty} \beta^j u(c_{t+j}). \quad (25)$$

where  $\beta = \frac{1}{(1+\rho)} < 1$  is the discount factor (not the MPC as earlier), and  $u(c)$  is a concave utility of consumption function:  $u'(c) > 0$ ,  $u''(c) < 0$ , with Inada conditions at the extremities  $(0, \infty)$ , which rules out zero consumption because  $u'(0) = \infty$ .

Consumer's are free to borrow and lend at the going interest rate subject to a "solvency condition" (a side-condition that rules out indefinitely large and growing consumer borrowing, analogous to the no-Ponzi scheme condition in the continuous time Ramsey-Cass-Koopmans environment)

$$\lim_{j \rightarrow \infty} \left( \frac{1}{R_{t+1+j}} \right)^j a_{t+j} = 0 \quad (26)$$

where  $R_{t+1} = (1 + r_{t+1})$  is the interest rate accumulation factor between periods  $t$  and  $t + 1$ , and initial wealth,  $a_t$ , is given.

The budget constraint is

$$a_{t+1} = R_{t+1} \cdot (a_t + qd_t - c_t). \quad (27)$$

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<sup>15</sup>The utility program runs to "infinity" but this does not change the sequential solution. (One can think of there being always a finite probability of living one more period, no matter what one's age.) Also remember that "consumption" strictly speaking means consumption *services*, roughly equivalent empirically to purchases of current services and non-durable goods plus the implicit value of the service flow per unit time from durable goods. A reflection of this concept is incorporated in fiscal systems that tax (a calibration of) the 'implicit rental value' of owner occupied housing.

Note that for a constant interest rate factor,  $R_t = R = (1 + r)$ , eq.(27) can be written

$$R \cdot a_t - a_{t+1} = R \cdot (c_t - qd_t) \quad (28)$$

$$a_t \cdot \left(1 - \frac{1}{R}L^{-1}\right) = (c_t - qd_t) \quad (29)$$

$$a_t = \sum_{j=0}^{\infty} \left(\frac{1}{R}\right)^j \cdot (c_{t+j} - qd_{t+j}) \quad (30)$$

$$\sum_{j=0}^{\infty} \left(\frac{1}{R}\right)^j c_{t+j} = \sum_{j=0}^{\infty} \left(\frac{1}{R}\right)^j qd_{t+j} + a_t. \quad (31)$$

Discounting by the constant interest rate, the intertemporal budget constraint therefore implies

$$PDV \text{ Consumption} = PDV \text{ Income} + \text{Current Wealth}.$$

Note that we are implicitly assuming that  $qd$  is generated by a stochastic process that the consumer is unable to control; we do not model labor supply decisions.

## 4.2 Intertemporal Optimality of Consumption

One approach to solving the consumer's problem (you have seen others) is to use dynamic programming (Bellman, 1957). First pose the value function,  $V_t(a_t)$ , defined as the present discounted value of expected utility of consumption evaluated along the optimal program:

$$\begin{aligned} V_t(a_t) &= \underset{\{c_{t+j} | [a_{t+1} = R_{t+1} \cdot (a_t + qd_t - c_t)]\}_{j=0}^{\infty}}{\text{Max}} E_t \sum_{j=0}^{\infty} \beta^j u(c_{t+j}) \\ &= \underset{\{c_{t+j} | [a_{t+1} = R_{t+1} \cdot (a_t + qd_t - c_t)]\}_{j=0}^{\infty}}{\text{Max}} \frac{E_t u(c_t)}{(1 - \beta L^{-1})} \end{aligned} \quad (32)$$

If we knew  $V_t(a_t)$ , the value function would imply the recursive relation known as a Bellman functional equation (Bellman's optimality condition). Multiplying through by  $(1 - \beta L^{-1})$  and preserving time  $t$  conditional expectations gives<sup>16</sup>

$$V_t(a_t) = \underset{\{c_t\}}{\text{Max}} \{u(c_t) + E_t [\beta \cdot V_{t+1}(a_{t+1})]\}, \quad (33)$$

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<sup>16</sup>Note that the time  $t$  conditional expectations operator is not affected by application of the forward operator  $L^{-1}$ . See the discussion in the 'Lag Operators' lecture.

which given the budget constraint could also be written

$$V_t(a_t) = \underset{\{c_t\}}{\text{Max}} \{u(c_t) + E_t [\beta \cdot V_{t+1}(R_{t+1} \cdot (a_t + qd_t - c_t))]\}. \quad (34)$$

Thus we have converted the multi-period problem of finding an infinite sequence of controls,  $c_{t+j}\}_{j=0}^{\infty}$ , to a two-period problem requiring one maximum each period that trades off consumption at time  $t$  for the value to the optimal program of more wealth at time  $t + 1$ . Hence the value function at time  $t$  equals the (known) utility of consumption at  $t$  plus the time  $t$  expectation of the discounted (unknown) value function at  $t + 1$ . It just says that the optimal utility of consumption program is obtained if one always takes the best action in the current period and pursues the optimal plan in the future.

### 4.2.1 The FOCs

The first-order necessary condition is

$$\frac{\partial V_t}{\partial c_t} = \frac{\partial u(c_t)}{\partial c_t} + E_t \frac{\partial [\beta \cdot V_{t+1}(a_{t+1})]}{\partial c_t} = 0 \quad (35)$$

$$= u'(c_t) + E_t \left\{ \beta \cdot \frac{\partial [V_{t+1}(a_{t+1})]}{\partial a_{t+1}} \cdot \frac{\partial a_{t+1}}{\partial c_t} \right\} = 0 \quad (36)$$

$$= u'(c_t) - E_t [\beta \cdot R_{t+1} \cdot V'_{t+1}(a_{t+1})] = 0$$

$$\Rightarrow u'(c_t) = E_t [\beta \cdot R_{t+1} \cdot V'_{t+1}(a_{t+1})]. \quad (37)$$

The FOC means that along the optimal program the marginal utility of current period consumption is equated to next period's expected discounted marginal value of financial wealth, magnified by the return to saving (deferred consumption).

We need to eliminate the unknown  $V'_{t+1}(a_{t+1})$ . The FOC implies that optimal consumption  $c_t^*$  is implicitly a function of current wealth. Using the Bellman equation

$$V_t(a_t) = \underset{\{c_t\}}{\text{Max}} \{u(c_t) + E_t [\beta \cdot V_{t+1}(R_{t+1} \cdot (a_t + qd_t - c_t))]\}$$

to find  $V'_t(a_t)$ , by the envelope theorem<sup>17</sup> we obtain

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<sup>17</sup>If we have a function of two variables  $f(a, c)$  such that for every  $a$  the maximum  $\text{Max}_c f(a, c)$  is achieved at an interior point  $c = c^*(a)$ , then the envelope theorem states

$$\frac{d}{da} \text{Max}_c f(a, c) = \frac{\partial}{\partial a} (c^*(a)).$$

$$\frac{\partial V_t}{\partial a_t} = \frac{\partial u(c_t)}{\partial c_t} \cdot \frac{\partial c_t}{\partial a_t} \Big|_{c_t=c_t^*} + E_t \left\{ \beta \cdot \frac{\partial [V_{t+1}(a_{t+1})]}{\partial a_{t+1}} \cdot \frac{\partial a_{t+1}}{\partial a_t} \Big|_{c_t=c_t^*} \right\} \quad (38)$$

$$- E_t \left\{ \beta \cdot \frac{\partial [V_{t+1}(a_{t+1})]}{\partial a_{t+1}} \cdot \frac{\partial a_{t+1}}{\partial a_t} \frac{\partial c_t}{\partial a_t} \Big|_{c_t=c_t^*} \right\}$$

$$\frac{\partial V_t}{\partial a_t} = u'(c_t) \cdot \frac{\partial c_t}{\partial a_t} \Big|_{c_t=c_t^*} \quad (39)$$

$$+ E_t \left\{ \beta \cdot R_{t+1} \cdot \frac{\partial [V_{t+1}(a_{t+1})]}{\partial a_{t+1}} \cdot \left( 1 - \frac{\partial c_t}{\partial a_t} \right) \Big|_{c_t=c_t^*} \right\}.$$

The FOC  $u'(c_t) = E_t [\beta \cdot R_{t+1} \cdot V'_{t+1}(a_{t+1})]$  implies that (39) simplifies to

$$\frac{\partial V_t}{\partial a_t} = E_t [\beta \cdot R_{t+1} \cdot V'_{t+1}(a_{t+1})] \quad (40)$$

$$= u'(c_t) \text{ by the consumption FOC.} \quad (41)$$

So along the optimal path the marginal value of time  $t$  financial wealth equals the marginal utility of time  $t$  consumption. It follows that

$$\frac{\partial V_{t+1}}{\partial a_{t+1}} = u'(c_{t+1}). \quad (42)$$

### 4.3 The Euler Equation

Substituting (42) into (37) gives the Euler equation

$$\begin{aligned} u'(c_t) &= E_t [\beta \cdot R_{t+1} \cdot u'(c_{t+1})] \\ &= E_t \frac{(1+r_{t+1})}{(1+\rho)} \cdot [u'(c_{t+1})]. \end{aligned} \quad (43)$$

Eq.(43) embodies an important result with which you are by now quite familiar, except in the present setting it is stochastic owing to uncertainty about future incomes.  $u'(c_t)$  gives the marginal utility cost of consuming a unit less of income today.  $\frac{(1+r_{t+1})}{(1+\rho)} \cdot E_t [u'(c_{t+1})]$  is the marginal utility gain tomorrow of deferring a unit of income's consumption today. It equals the discounted marginal utility magnified by the one-period return to saving. By the backward induction logic of the Bellman equation (dynamic programming), at an optimum these relations must hold at every period, otherwise an intertemporal arbitrage opportunity would exist that would offer a gain to the utility of consumption program.

To nail the last point, consider two perturbations to the Euler equation:

(i). Suppose  $u'(c_t) > E_t \frac{(1+r_{t+1})}{(1+\rho)} \cdot [u'(c_{t+1})]$ , then *raise* consumption by  $\text{€}$  today and *lower* consumption by  $\text{€} \cdot (1+r_{t+1})$  tomorrow to harvest a net utility gain:<sup>18</sup>

$$+u'(c_t) - E_t \frac{(1+r_{t+1})}{(1+\rho)} \cdot [u'(c_{t+1})] > 0. \quad (44)$$

(ii) Suppose  $u'(c_t) < E_t \frac{(1+r_{t+1})}{(1+\rho)} \cdot [u'(c_{t+1})]$ , then *lower* consumption today by  $\text{€}$  today and *raise* consumption by  $\text{€} \cdot (1+r_{t+1})$  tomorrow to harvest a net utility gain:

$$-u'(c_t) + E_t \frac{(1+r_{t+1})}{(1+\rho)} \cdot [u'(c_{t+1})] > 0. \quad (45)$$

Note that the first perturbation is always possible only if there is no liquidity constraint. No one ever has zero consumption, so the second perturbation should always be feasible. To say much more than this we must specify  $u(c)$ .

## 5 Hall's Random Walk Result

Hall's (JPE, 1978) famous random walk result hinged on assuming quadratic utility which, as already noted, has a "certainty equivalence" property:

$$u(c_t) = -\frac{1}{2} (\bar{c} - c_t)^2 \quad (46)$$

where  $\bar{c}$  is a "saturation level" of consumption (think of it as so high as never to be achieved).<sup>19</sup> Marginal utility is therefore linear,  $u'(c) = (\bar{c} - c_t) > 0$ , the second derivative is negative satisfying concavity,  $u''(c) = -1$ , and the third derivative is zero, implying certainty equivalence in consumption behavior. With quadratic utility the Euler equation  $u'(c_t) = E_t \frac{(1+r_{t+1})}{(1+\rho)} \cdot [u'(c_{t+1})]$  is

$$(\bar{c} - c_t) = E_t \left[ \frac{(1+r_{t+1})}{(1+\rho)} \cdot (\bar{c} - c_{t+1}) \right] \quad (47)$$

$$c_t = \left[ 1 - E_t \frac{(1+r_{t+1})}{(1+\rho)} \right] \cdot \bar{c} + E_t \left[ \frac{(1+r_{t+1})}{(1+\rho)} \cdot c_{t+1} \right].$$

<sup>18</sup>Keep in mind  $u''(c) < 0$ , so raising  $c$  decreases  $u'(c)$ , and conversely.

<sup>19</sup>Hall used de-trended aggregate real per capita expenditure on non-durable goods and services for  $c$  in tests of the PIH with National Accounts data.

If the interest rate is constant at  $r$  we have

$$c_t = \left[ 1 - \frac{(1+r)}{(1+\rho)} \right] \cdot \bar{c} + \frac{(1+r)}{(1+\rho)} \cdot E_t c_{t+1}, \quad (48)$$

and if  $\rho = r$ , we obtain at every period

$$c_t = E_t c_{t+1} \quad (49)$$

$$c_{t+1} = E_t c_{t+2} \quad (50)$$

$$\begin{aligned} & \cdot \\ c_{t+j-1} &= E_t c_{t+j} \end{aligned} \quad (51)$$

so that

$$c_t = E_t c_{t+1} = E_t c_{t+2} = \dots = E_t c_{t+j}, \quad \text{for all } j > 0. \quad (52)$$

Hence consumption is a "martingale" which implies directly the famous random walk regression relation:

$$c_{t+1} = c_t + e_{t+1} \quad (53)$$

where  $e_{t+1} = (c_{t+1} - E_t c_{t+1}) \sim 0$ , *white*, a requirement of rationality of expectations.  $\Delta c_{t+1}$  therefore cannot be predicted ex-ante, that is, cannot be forecasted with any information available at time  $t$ .

## 5.1 Retrieving Friedman's PIH from Hall

Hall's setup has all the requirements that I claimed earlier would give Friedman's permanent income model; specifically it will yield the equation:

$$c_t = qd_t^P = \frac{r}{(1+r)} \left[ E_t \sum_{j=0}^{\infty} \frac{qd_{t+j}}{(1+r)^j} + a_t \right].$$

Let me now substantiate the claim. Recall that for  $r$  constant the budget constraint implies the relation

$$\sum_{j=0}^{\infty} \left( \frac{1}{R} \right)^j c_{t+j} = \sum_{j=0}^{\infty} \left( \frac{1}{R} \right)^j qd_{t+j} + a_t.$$

Taking the time  $t$  expectation of this constraint and expressing it using the first-order forward polynomial, we have

$$E_t \left( \frac{c_t}{1 - \frac{1}{R}L^{-1}} \right) = E_t \frac{qd_t}{1 - \frac{1}{R}L^{-1}} + a_t. \quad (54)$$

We know from the martingale/random walk result that  $E_t c_{t+j} = c_t$  at all  $j > 0$ , so the left-side of eq.(54) is<sup>20</sup>

$$E_t \left( \frac{c_t}{1 - \frac{1}{R}L^{-1}} \right) = \left( \frac{1}{1 - \frac{1}{R}} \right) c_t = \frac{(1+r)}{r} c_t. \quad (55)$$

Substituting (55) into (54) gives

$$\frac{(1+r)}{r} c_t = E_t \frac{qd_t}{1 - \frac{1}{R}L^{-1}} + a_t, \quad (56)$$

and, therefore, we arrive at the consumption function

$$\begin{aligned} c_t &= \frac{r}{(1+r)} \left[ E_t \frac{qd_t}{1 - \frac{1}{R}L^{-1}} + a_t \right] \\ &= qd_t^P \end{aligned} \quad (57)$$

which is the result that was claimed.<sup>21</sup>

Consumption is therefore an annuity of expected lifetime (infinite in this setup) wealth – which is the way Franco Modigliani framed the problem – with lifetime wealth given by the sum of current wealth and the expected PDV of disposable income – and this equals permanent income, which is the way Milton Friedman framed the problem. You might convince yourself further that the consumption level of (57) could indeed be sustained in perpetuity by, for example, assuming that  $qd_{t+j}$  is held at some income level  $\bar{q}\bar{d}$ , and checking that the result holds.

Earlier we concluded that  $\Delta c_{t+1} = e_{t+1}$ , where  $e_{t+1} = (c_{t+1} - E_t c_{t+1}) \sim 0$ , *white*. Let's see what this turns out to mean given eq.(57).

$$c_{t+1} - c_t = [qd_{t+1}^P - qd_t^P] \quad (58)$$

$$c_{t+1} - c_t = \frac{r}{(1+r)} \cdot \left[ \begin{array}{l} (E_{t+1} PDV(qd_{t+1}) + a_{t+1}) \\ - (E_t PDV(qd_t) + a_t) \end{array} \right]. \quad (59)$$

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<sup>20</sup>Recall the result for convergent geometric series in the 'Lag Algebra' lecture. Note that if the consumer's optimization program ran from  $c_{t+j}$ ,  $j = 0$  to  $j = T - t$ , then

$$\sum_{j=0}^{T-t} \left( \frac{1}{R} \right)^j c_t = \left[ \frac{(1+r)}{r} - \frac{(1+r)^{-(T-t)}}{r} \right] c_t.$$

<sup>21</sup>Note also that Hall's result is for the "representative agent". See Clarida, QJE, 1991 for an investigation of aggregation over cohorts of agents in working life and retirement.

But we know that consumption is a martingale:  $E_t c_{t+1} = c_t$ . So we can replace the second right-side expression within brackets with the time  $t$  expectation of the first right-side expression within brackets:

$$c_{t+1} - c_t = \frac{r}{(1+r)} \cdot \left[ \begin{array}{l} (E_{t+1} PDV(qd_{t+1}) + a_{t+1}) \\ -E_t (E_{t+1} PDV(qd_{t+1}) + a_{t+1}) \end{array} \right]. \quad (60)$$

By the law of iterated expectations  $E_t [E_{t+n}] = E_t$ ,  $n \geq 0$ . [*Side-board discussion of this point*] Hence we have

$$c_{t+1} - c_t = \frac{r}{(1+r)} \cdot \left[ \begin{array}{l} (E_{t+1} PDV(qd_{t+1}) + a_{t+1}) \\ -(E_t PDV(qd_{t+1}) + E_t a_{t+1}) \end{array} \right] \quad (61)$$

$$c_{t+1} - c_t = \frac{r}{(1+r)} \cdot [(a_{t+1} - E_t a_{t+1}) + (E_{t+1} - E_t) PDV(qd_{t+1})]. \quad (62)$$

From the budget constraint we know that when  $r$  is constant wealth accumulates by  $a_{t+1} = R \cdot (a_t + qd_t - c_t)$ . So the first right-side expression within brackets in eq.(62) is

$$\begin{aligned} (a_{t+1} - E_t a_{t+1}) &= R \cdot (a_t + qd_t - c_t) - E_t [R \cdot (a_t + qd_t - c_t)] \\ &= 0 \end{aligned} \quad (63)$$

because all time  $t$  variables are known at time  $t$ . Changes in consumption are therefore driven by

$$\begin{aligned} c_{t+1} - c_t &= \frac{r}{(1+r)} \cdot [(E_{t+1} - E_t) PDV(qd_{t+1})] \\ &= c_{t+1} - E_t c_{t+1}, \\ &= e_{t+1} \sim 0, \text{ white,} \end{aligned}$$

that is,  $\Delta c_{t+1}$  is proportional to (is an annuity of) revisions to expectations of the PDV of disposable income (annuitized income "surprises"). By rational expectations such revisions must have mean zero and serial independence, otherwise agents would have overlooked something systematic when calibrating their permanent income.

Next note that different specifications of utility yield different Euler equations and, consequently, different implications for consumption's stochastic properties. You should thoroughly understand this point. Yet second differentiable functions are usually approximately quadratic (which we know from Taylor series expansions), so the Hall result is at least approximately true for many utility functions.

## 6 Short-Run Fiscal Policy Implications of the Stochastic PIH with Rational Expectations

Only innovations or 'surprises' to fiscal policy and, hence, to permanent income (inherently unforecastable. changes to the PDV of disposable income) can affect consumption behavior, because all other fiscal effects are already incorporated into permanent income and current consumption. It follows that Fiscal authorities have no capacity to affect consumption, and through consumption the macroeconomy, via predictable or systematic fiscal changes, unless of course such changes in the fisc are in reaction to economic conditions that are themselves unforecastable. (In a sense, this could in fact be the case; real output, for example, is well characterized by a random walk with drift.) However this may be, the main message of the stochastic PIH is that only unanticipated changes to taxes or transfers can move current consumption, because such changes are innovations to permanent income (or, equivalently innovations to the annuity value of wealth). This somewhat nihilistic result is perhaps the main policy message of the whole rational expectations revolution in macroeconomic theory, which also (in fact, more commonly) is targeted on monetary policy activism. In a few weeks you will be hearing more about this in the lecture on 'Lucasion' aggregate supply models. In the next lecture we will explore how government spending on goods and services might effect consumption.

### Lecture References

Blanchard and Fisher, Lectures in Macroeconomics, MIT 1989, chapter 6.  
Ljungqvist and Sargent, Recursive Macroeconomic Theory, MIT 2000

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