

Dynamic General Macroeconomic Equilibrium

This lecture runs through a bare bones model of dynamic general macroeconomic equilibrium. During subsequent lectures the constituents of the model will receive sustained attention.

1 The state of the world

The state of the world is standard neoclassical in a Ramsey-Cass-Koopmans setting. The economy is composed of numerous, identical price taking households and firms. At every point in time three goods are traded: labor services N , capital K , and a single final output good, Q , which can be allocated to investment, I , or consumption, C . There is no government sector. Time is discrete and the economy is deterministic.

Output production is Cobb-Douglas:

$$Q_t = A_t K_t^\alpha N_t^{1-\alpha} \quad (1)$$

where A is the exogenously given state of technology, and $0 < \alpha < 1$.

The accumulation equation for the stock of capital (there is no depreciation) is:

$$\begin{aligned} K_{t+1} &= (Q_t - C_t) + K_t \\ &= I_t + K_t. \end{aligned} \quad (2)$$

Households are infinitely lived ("dynasties"), and they value expected consumption and leisure streams according to the utility function:

$$U = E_{t=0} \sum_{t=0}^{\infty} \beta^t \left[\ln C_t - \frac{(N_t^s)^{1+\phi}}{1+\phi} \right] \quad (3)$$

where $\beta = \frac{1}{(1+\rho)}$ is the discount factor, $\rho > 0$ is the rate of time preference, C is consumption, N^s is household labor supply to firms, and $\phi > 0$ is the elasticity of

the marginal utility of time, that is

$$\begin{aligned}\phi &= \frac{\partial U'(N_t^s)}{\partial N_t^s} \cdot \frac{N_t^s}{U'(N_t^s)} \\ &= \left(-\phi\beta^t (N_t^s)^{\phi-1}\right) \cdot \frac{N_t^s}{-\beta^t (N_t^s)^\phi}\end{aligned}$$

The corresponding elasticity of marginal utility of consumption is minus unity (a notable feature of log utility):

$$\begin{aligned}-1 &= \frac{\partial U'(C_t)}{\partial C_t} \cdot \frac{C_t}{U'(C_t)} \\ &= -\beta^t C_t^{-2} \cdot \frac{C_t}{\beta^t C_t^{-1}}.\end{aligned}$$

As we shall see ahead, log utility of consumption implies that households are willing to substitute intertemporally point-for-point with deviations of the rate of time preference from the rate at which assets accumulate (the interest rate).

2 The Producer's Program

Firms choose demand for capital and labor at every point in time to maximize a one-period real profit function, π_t , taking the real wage, w , and the cost of capital, r , as given:

$$\begin{aligned}\pi_t &= Q_t - w_t N_t^d - r_t K_t^d \\ &= A_t K_t^\alpha N_t^{1-\alpha} - w_t N_t^d - r_t K_t^d.\end{aligned}\tag{4}$$

The FOC for labor demand is

$$\frac{\partial \pi_t}{\partial N_t^d} = 0 = (1 - \alpha) A_t (K_t^d)^\alpha (N_t^d)^{-\alpha} - w_t\tag{5}$$

so that the given wage equals the marginal product of labor:

$$w_t = (1 - \alpha) A_t (K_t^d)^\alpha (N_t^d)^{-\alpha}.\tag{6}$$

The FOC for capital demand is

$$\frac{\partial \pi_t}{\partial K_t^d} = 0 = \alpha A_t (K_t^d)^{\alpha-1} (N_t^d)^{1-\alpha} - r_t\tag{7}$$

so that the given cost of capital is equated to the marginal product of capital:

$$r_t = \alpha A_t (K_t^d)^{\alpha-1} (N_t^d)^{1-\alpha}.\tag{8}$$

3 The Consumer's Program

Households own the capital stock and are initially endowed with K_0 units along with \bar{N} units of time per period. By assumption \bar{N} is large enough not to be binding. (As in the case of firms, aggregation issues are fudged over and I proceed as if the representative household subsumes the entire household sector. I return to this matter in subsequent lectures.) Households choose C, N^s, K^s and I to maximize the expected intertemporal utility function

$$Max_{\{C_t, N_t^s, K_t^s, I_t\}_{t=0}^{\infty}} E_{t=0} \sum_{t=0}^{\infty} \beta^t \left[\ln C_t - \frac{(N_t^s)^{1+\phi}}{1+\phi} \right] \quad (9)$$

subject a time constraint

$$N_t^s \leq \bar{N}.$$

and a budget constraint

$$\begin{aligned} I_t + C_t &= (K_{t+1}^s - K_t^s) + C_t \\ &= r_t K_t^s + w_t N_t^s = Q_t \end{aligned}$$

Note that the second line of the budget constraint above follows from constant returns to scale (Cobb-Douglas) production; factor payments exhaust output:

$$Q_K K + Q_L L = Q.$$

I will proceed here (but not in subsequent lectures) as if agents had perfect foresight, that is, $E_{t=0} X_t = X_t$ for any variable X at any time $t \geq 0$. The conditional expectations operator is therefore without effect (arm-waving).

4 Optimal Consumption and Labor Supply

Let λ_t denote the Lagrangian multiplier on the time t budget constraint. The household's sequential decision problem can be posed as:

$$Max_{\{C_t, N_t^s, K_{t+1}^s\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[\ln C_t - \frac{(N_t^s)^{1+\phi}}{1+\phi} + \lambda_t \cdot [r_t K_t^s + w_t N_t^s - (K_{t+1}^s - K_t^s) - C_t] \right] \quad (10)$$

subject to the transversality condition

$$\lim_{t \rightarrow \infty} \beta^t \lambda_t K_t = 0$$

and the time constraint

$$N_t^s \leq \bar{N}.$$

Recall that there is no depreciation of capital and therefore in the household budget constraint $I_t = (K_{t+1}^s - K_t^s)$. The FOC for time t choice of consumption is

$$\beta^t \left(\frac{1}{C_t} - \lambda_t \right) = 0 \quad (11)$$

yielding

$$\lambda_t = \frac{1}{C_t}. \quad (12)$$

The value of the Lagrangian multiplier therefore equals the marginal utility of consumption at each period, since $U'(C_t) = \frac{1}{C_t}$.

The FOC for choice of next period's capital (K_t^s is given at each decision period), K_{t+1}^s , is:

$$-\beta^t \lambda_t + \beta^{t+1} \lambda_{t+1} (r_{t+1} + 1) = 0. \quad (13)$$

Note that this FOC is derived by evaluating (11) at periods t and $t + 1$. The K_{t+1}^s FOC may be rearranged to obtain:

$$\begin{aligned} \beta^t \lambda_t &= \beta^{t+1} \lambda_{t+1} (1 + r_{t+1}) \\ \lambda_t &= \frac{\beta^{t+1}}{\beta^t} \lambda_{t+1} (1 + r_{t+1}), \end{aligned} \quad (14)$$

which since $\beta = \frac{1}{(1+\rho)}$ gives

$$\lambda_t = \lambda_{t+1} \frac{(1 + r_{t+1})}{(1 + \rho)}. \quad (15)$$

The implication is that the household equates the marginal utility of an extra unit of consumption today (λ_t) to the discounted marginal utility of consuming next period the marginal product of this consumption unit, $(\lambda_{t+1} \cdot \frac{(1+r_{t+1})}{(1+\rho)})$.

The FOCs for C_t and K_{t+1} combined give the *Euler equation for optimal intertemporal household consumption behavior* (often called the Keynes-Ramsey rule). We already know

$$\lambda_t = \frac{1}{C_t}$$

$$\lambda_{t+1} = \frac{1}{C_{t+1}}$$

$$\lambda_t = \lambda_{t+1} \frac{(1 + r_{t+1})}{(1 + \rho)}.$$

Hence after substitution for the Lagrangians we have

$$\frac{1}{C_t} = \frac{1}{C_{t+1}} \frac{(1 + r_{t+1})}{(1 + \rho)} \quad (16)$$

which implies

$$\frac{C_{t+1}}{C_t} = \frac{(1 + r_{t+1})}{(1 + \rho)}. \quad (17)$$

Since $\ln \left(\frac{(1+r_{t+1})}{(1+\rho)} \right) = [\ln(1 + r_{t+1}) - \ln(1 + \rho)] \cong (r_{t+1} - \rho)$ when r and ρ are "small"¹, we have at all $t > 0$

$$\ln C_{t+1} - \ln C_t = (r_{t+1} - \rho) \quad (18)$$

$$\ln C_{t+1} (1 - L) = (r_{t+1} - \rho).$$

After multiplying the left- and right-sides of the second line of (18) by $(1 + L + L^2 + \dots + L^t)$, we obtain a solution for log consumption at $t + 1$ in terms of initial condition $\ln C_0$:

$$\ln C_{t+1} (1 - L) \cdot (1 + L + L^2 + \dots + L^t) = [(r_{t+1} - \rho) \cdot (1 + L + L^2 + \dots + L^t)] \quad (19)$$

$$\begin{aligned} \ln C_{t+1} (1 - L^{t+1}) &= \sum_{j=0}^t L^j (r_{t+1} - \rho) \\ &= \sum_{j=0}^t r_{t+1-j} - (t + 1) \rho. \end{aligned} \quad (20)$$

If returns to saving are constant at, say, rate \bar{r} , we obtain:

$$\ln C_{t+1} = \ln C_0 + (t + 1) \cdot (\bar{r} - \rho). \quad (21)$$

¹Taylor's approximation is

$$\begin{aligned} \ln(1 + x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \\ 0 &< x < 1. \end{aligned}$$

For small x the first term is very accurate.

The optimal consumption time path is therefore upward sloping from period $t = 0$ forward when the interest rate exceeds the rate of time preference. Households defer consumption today in favor of higher consumption later on. We shall see this famous equation in continuous time and in more general context when I revisit the Ramsey-Cass-Koopmans model during the growth theory lectures.

Returning to the household's program,

$$\text{Max}_{\{C_t, N_t^s, K_{t+1}^s\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[\lambda_t \cdot \left[r_t K_t^s + w_t N_t^s - (K_{t+1}^s - K_t^s) - C_t \right] \right],$$

the FOC for choice of labor supply is:

$$\beta^t \left(- (N_t^s)^\phi + \lambda_t w_t \right) = 0 \quad (22)$$

implying that

$$(N_t^s)^\phi = \lambda_t w_t \quad (23)$$

$$U'(N_t^s) = w_t U'(C_t). \quad (24)$$

Hence labor is supplied to market activity such that the marginal (dis)utility of labor is equated to the wage *weighted by* the value to time t marginal utility of consumption opportunities that are afforded by the wage income obtained from an extra unit of labor supply. Put another way, at the optimum the marginal disutility of working must equal the marginal benefit of working, which is the value of the wage in units of marginal consumption.

We can now derive a dynamic labor supply equation. We know from (12) and (23) that

$$(N_t^s)^\phi = \frac{1}{C_t} w_t, \quad (25)$$

and we know from the intertemporal symmetry of the sequential solution that

$$(N_{t+1}^s)^\phi = \frac{1}{C_{t+1}} w_{t+1}. \quad (26)$$

Hence

$$\frac{N_{t+1}^s}{N_t^s} = \left[\frac{w_{t+1}}{w_t} \left(\frac{C_{t+1}}{C_t} \right)^{-1} \right]^{\frac{1}{\phi}}. \quad (27)$$

Using the result in (17), $\frac{C_{t+1}}{C_t} = \frac{(1+r_{t+1})}{(1+\rho)}$, we find

$$\frac{N_{t+1}^s}{N_t^s} = \left[\frac{w_{t+1}}{w_t} \left(\frac{(1+r_{t+1})}{(1+\rho)} \right)^{-1} \right]^{\frac{1}{\phi}} \quad (28)$$

and taking logs we obtain

$$(\ln N_{t+1}^s - \ln N_t^s) = (1/\phi) \cdot [(\ln w_{t+1} - \ln w_t) - (r_{t+1} - \rho)] \quad (29)$$

where I again make use of the fact that $\ln(1+r_{t+1}) \cong r_{t+1}$, and $\ln(1+\rho) \cong \rho$.

An implication of this result is that for given growth of wages, households choose a downward sloping time path for market labor supply when the interest rate exceeds the discount rate. In other words, when $r > \rho$ market work today is higher, and leisure is deferred.

Equation (28) implies

$$\ln N_{t+1}^s (1-L) = (1/\phi) (\ln w_{t+1} - \ln w_t) - (1/\phi) (r_{t+1} - \rho) \quad (30)$$

which after multiplication by $(1+L+L^2+\dots+L^t)$ gives a solution for the level of log labor supply at period $t+1$ from initial condition $\ln N_0^s$:

$$\begin{aligned} \ln N_{t+1}^s (1-L) \cdot (1+L+L^2+\dots+L^t) &= (1/\phi) \cdot \sum_{j=0}^t L^j \left[\begin{array}{c} (\ln w_{t+1} - \ln w_t) \\ - (r_{t+1} - \rho) \end{array} \right] \\ \ln N_{t+1}^s (1-L^{t+1}) &= (1/\phi) \cdot \sum_{j=0}^t L^j \left[\begin{array}{c} (\ln w_{t+1} - \ln w_t) \\ - (r_{t+1} - \rho) \end{array} \right] \\ \ln N_{t+1}^s &= \ln N_0^s + (1/\phi) \cdot \left[\begin{array}{c} (\ln w_{t+1} - \ln w_0) \\ - \left(\sum_{j=0}^t r_{t+1-j} - (t+1) \cdot \rho \right) \end{array} \right]. \end{aligned} \quad (31)$$

For r_{t+1} equal to \bar{r} at all t , the solution for log labor supply therefore is

$$\ln N_{t+1}^s = \ln N_0^s + (1/\phi) \cdot [(\ln w_{t+1} - \ln w_0) - (t+1) \cdot (\bar{r} - \rho)] \quad (32)$$

which, as already noted, means that for given wage changes the household's optimal labor supply is downward sloping over time (and, as shown earlier, optimal consumption is upward sloping) when the interest rate exceeds the rate of time preference. Hence, high real interest rates tend to produce a world of relatively low current consumption (high current saving and investment), and relatively high current propensity to supply market work.

5 Equilibrium Requirements and Outcomes

Given K_0 , a competitive equilibrium requires:

- $Q_t = C_t + I_t$, the goods market clears
- $N_t^s = N_t^d$, the labor market clears
- $K_t^s = K_t^d$, the capital market clears
- the sequences $\{K_t^d, N_t^d\}_{t=0}^\infty$ solve the producer's program
- the sequences $\{C_t, N_t^s, K_t^s, I_t\}_{t=0}^\infty$ solve the consumer's program.

Using the firm's demand for labor relation in (6), which equates the wage to the marginal product of labor deployed, $w_t = (1 - \alpha) A_t (K_t^d)^\alpha (N_t^d)^{-\alpha}$, in combination with the household's optimal labor supply relations in (24)- (25), $(N_t^s)^\phi = \lambda_t w_t = \frac{1}{C_t} w_t$, and the requirement of competitive equilibrium, $N_t^s = N_t^d$, we can find the employment equation :

$$\begin{aligned} N_t &= \left(\frac{1}{C_t} w_t \right)^{\frac{1}{\phi}} \\ &= C_t^{-\frac{1}{\phi}} \cdot [(1 - \alpha) A_t (K_t)^\alpha (N_t)^{-\alpha}]^{\frac{1}{\phi}} \end{aligned} \quad (33)$$

$$\begin{aligned} N_t \cdot N_t^{\frac{\alpha}{\phi}} &= C_t^{-\frac{1}{\phi}} \cdot [(1 - \alpha) A_t (K_t)^\alpha]^{\frac{1}{\phi}} \\ (N_t)^{\frac{(\phi+\alpha)}{\phi}} &= C_t^{-\frac{1}{\phi}} \cdot [(1 - \alpha) A_t (K_t)^\alpha]^{\frac{1}{\phi}} \end{aligned} \quad (34)$$

$$\begin{aligned} N_t &= \left\{ C_t^{-\frac{1}{\phi}} [(1 - \alpha) A_t (K_t)^\alpha]^{\frac{1}{\phi}} \right\}^{\frac{\phi}{(\phi+\alpha)}} \\ &= (1 - \alpha)^{\frac{1}{(\phi+\alpha)}} A_t^{\frac{1}{(\phi+\alpha)}} C_t^{-\frac{1}{(\phi+\alpha)}} K_t^{\frac{\alpha}{(\phi+\alpha)}}. \end{aligned} \quad (35)$$

For a given state of technology, A_t , the (conditional) steady-states for the bare bones setup now follow directly. From the household's Euler equation in (17),

$$\frac{C_{t+1}}{C_t} = \frac{(1 + r_{t+1})}{(1 + \rho)},$$

we see that steady-state consumption, $C_{t+1} = C_t = C^*$ (or $\frac{C_{t+1}}{C_t} = 1$) requires $r^* = \rho$. From the capital accumulation equation in (2), it follows that at (conditional) equilibrium the capital stock satisfies $K_{t+1} = K_t = K^*|A_t$, and so investment, I , will be zero. (Recall there is no depreciation of the capital stock.) Hence equilibrium consumption equals equilibrium output (conditional on the state of technology), $C^* = Q^* = A_t K^{*\alpha} N^{*1-\alpha}$.

To obtain *steady-state employment*, substitute the steady-state condition $C^* = Q^*$ into the equilibrium employment equation in (35) above:

$$\begin{aligned}
N^* &= (1 - \alpha)^{\frac{1}{(\phi + \alpha)}} A_t^{\frac{1}{(\phi + \alpha)}} C^{* - \frac{1}{(\phi + \alpha)}} K^{* \frac{\alpha}{(\phi + \alpha)}} \\
N^* &= (1 - \alpha)^{\frac{1}{(\phi + \alpha)}} A_t^{\frac{1}{(\phi + \alpha)}} \left(A_t K^{*\alpha} N^{*1 - \alpha} \right)^{-\frac{1}{(\phi + \alpha)}} K^{* \frac{\alpha}{(\phi + \alpha)}} \\
N^* \cdot (N^*)^{\frac{(1 - \alpha)}{(\phi + \alpha)}} &= (1 - \alpha)^{\frac{1}{(\phi + \alpha)}} \\
N^{* \frac{(\phi + 1)}{\phi + \alpha}} &= (1 - \alpha)^{\frac{1}{(\phi + \alpha)}} \\
N^* &= (1 - \alpha)^{\frac{1}{(1 + \phi)}}. \tag{36}
\end{aligned}$$

Notice that in general equilibrium steady-state employment (labor input to production)

(i) rises with labor's share of income, $\frac{Q'(N) \cdot N}{Q} = (1 - \alpha)$:

$$\frac{dN^*}{d(1 - \alpha)} \equiv \frac{d \left[(1 - \alpha)^{\frac{1}{(1 + \phi)}} \right]}{d(1 - \alpha)} = \frac{1}{(1 + \phi)} \cdot \left[(1 - \alpha)^{\frac{1}{1 + \phi} - 1} \right] > 0$$

(ii) rises with the elasticity of the marginal value of time, ϕ :

Since $\frac{d \ln N^*}{d\phi} = \frac{1}{N^*} \cdot \frac{dN^*}{d\phi}$,

$$\begin{aligned}
\frac{dN^*}{d\phi} &= \frac{d \ln N^*}{d\phi} \cdot N^* = \frac{d \left[\frac{1}{(1 + \phi)} \ln(1 - \alpha) \right]}{d\phi} \cdot (1 - \alpha)^{\frac{1}{(1 + \phi)}} \\
&= -\frac{\ln(1 - \alpha)}{(1 + \phi)^2} (1 - \alpha)^{\frac{1}{1 + \phi}} > 0
\end{aligned}$$

given $(0 < \alpha < 1)$.

(Note: If the above result seems counterintuitive, remember that ϕ is the elasticity of marginal utility of N^s , that is, the proportional change in marginal utility, $U'(N^s)$, generated by a proportional change in N^s , as shown earlier.)

and

(iii) is not affected by the technology-productivity term A , which is necessary for the existence of a steady state when A_t grows (a topic to be pursued at length in the growth theory lectures). This result follows from the specification of log utility; the income and substitution effects of rising A_t (rising productivity and real wages) cancel out.

The *steady-state stock of capital* may be obtained by using the equilibrium condition $r^* = \rho$ and substituting N^* into the demand for capital in (8):

$$r^* = \rho = \alpha A_t K^{*(\alpha-1)} (N^*)^{1-\alpha} \quad (37)$$

$$\rho = \alpha A_t K^{*(\alpha-1)} \left((1-\alpha)^{\frac{1}{1+\phi}} \right)^{1-\alpha} \quad (38)$$

which implies that the technology-conditional equilibrium steady-state capital stock is

$$\begin{aligned} K^{*(1-\alpha)} \cdot \rho &= \alpha A_t \left((1-\alpha)^{\frac{1}{1+\phi}} \right)^{1-\alpha} \\ K^* &= \left(\frac{\alpha A_t}{\rho} \right)^{\frac{1}{1-\alpha}} (1-\alpha)^{\frac{1}{1+\phi}} \\ K^*|A_t &= \left(\frac{\alpha A_t (1-\alpha)^{\frac{1-\alpha}{1+\phi}}}{\rho} \right)^{\frac{1}{1-\alpha}}. \end{aligned} \quad (39)$$

You should convince yourself that steady-state stock of capital

- (i) rises with capital's share, α
- (ii) falls with the rate of time preference, ρ
- (iii) rises with the elasticity of the marginal value of time, ϕ ,

and of course

- (iv) increases with technological progress (productivity), A .

Finally, technology-conditional *equilibrium steady-state consumption and income* are:

$$\begin{aligned} C^*|A_t &= Q^*|A_t = A_t K^{*\alpha} N^{*1-\alpha} \\ &= A_t \left(\frac{\alpha A_t (1-\alpha)^{\frac{1-\alpha}{1+\phi}}}{\rho} \right)^{\frac{\alpha}{1-\alpha}} (1-\alpha)^{\frac{1-\alpha}{1+\phi}}. \end{aligned} \quad (40)$$

Comparative statics follow from the results for K^* and N^* .

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