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Some Growth Theory Empirics

In this lecture I cover just a small subset of growth theory empirics pertaining to the wealth and poverty of nations.

1 Implications of the Neoclassical Model for Relative Incomes

1.1 *The Bare Bones Neoclassical Regime*

I shall focus on income per worker ($\widetilde{q} \equiv Q/N$) rather than income per "effective worker" ($q \equiv Q/AN$) because the former holds greater practical interest. As pointed out in previous lectures, we cannot say very much about income levels unless we commit to a production technology.

The workhorse empirical neoclassical specification is Cobb-Douglas (often augmented to break-out human capital), and that is what I will work with.

For bare bones Cobb-Douglas production we already know that steady-state incomes per worker are[1]

$$\widetilde{q}^{*} | \left(k = k^{*}\right) = A_{t} \cdot k^{*\alpha} \tag{1}$$

[1] Or, if the Cobb-Douglas is written

$$Q = K^{\alpha} \left(AN\right)^{1-\alpha}$$

and capital also is expressed per worker,

$$\widetilde{q}^{*} | \left(k = k^{*}\right) = A_{t}^{(1-\alpha)} \widetilde{k}^{*\alpha}.$$

Note that Barro and Sala-i-Martin usually work with Cobb-Douglas in the form

$$Q = AK^{\alpha}N^{1-\alpha}$$

which would give

$$\widetilde{q}^{*} | \left(k = k^{*}\right) = A_{t}^{\frac{1}{1-\alpha}} \cdot \left[\frac{s}{(\delta + g + n)}\right]^{\frac{\alpha}{(1-\alpha)}}$$

as the convergent level of output per worker.

$$\widetilde{q}^{*}|\,(k=k^{*})=A_{t}\cdot\left[\frac{s}{(\delta+g+n)}\right]^{\frac{\alpha}{(1-\alpha)}} \tag{2}$$

with $0<\alpha<1$ and assuming $\dot{A}=gA$ and $\dot{N}=nN$.

The relative income of the world's leading economies (or the leading economy, usually taken to be the US of A), $\widetilde{q}_{w}^{*}$, as compared to any other, $\widetilde{q}_{j}^{*}$, therefore is

$$\begin{aligned}\frac{\widetilde{q}_{w}^{*}}{\widetilde{q}_{j}^{*}} &= \left\{\frac{\left[\frac{s_{w}}{(\delta+g+n_{w})}\right]}{\left[\frac{s_{j}}{(\delta+g+n_{j})}\right]}\right\}^{\frac{\alpha}{(1-\alpha)}} \\ &= \left[\frac{s_{w}}{s_{j}}\cdot\frac{(\delta+g+n_{j})}{(\delta+g+n_{w})}\right]^{\frac{\alpha}{(1-\alpha)}}\end{aligned} \tag{3}$$

where I assume that the depreciation rate and (for the moment) that the rate of technology growth are common (the standard neoclassical assumption).

We can assume that $(\delta+g)$, measured on an annual basis, is probably not less than than around 0.075, and that labor force growth rates might run as high as 0.05 in poor countries and as low as 0.0 in rich ("leading-economy") countries. The poor-to-rich ratio of these parameters in the second right-side term in brackets in eq.(3) therefore should be no greater than about 1.7.

The ratio of output per worker in the most and least productive 5 percent of national economies is about 30 (Summers and Heston, World Bank data). At the extremes the ratio runs as high as 100 or greater. At $\alpha=1/3$, the typical income share of physical capital in national accounts, eq.(3) implies that saving rates in leading-economy nations would have to be around 540 times higher than in poor countries in order to account for the thirty-fold development gap observed in contemporary data. 540 is way too big. Hence, with a traditional conception of capital, the bare bones Cobb-Douglas neoclassical model cannot possibly account for the wealth and poverty of nations.

Mankiw, Romer and Weil (QJE, 1992) – henceforth 'MRW' – evaluate the bare bones Cobb-Douglas model by running a cross-section regression based on eq.(3) for log GDP per working age person; a specification which of course assumes that countries are at, or are nearly at, steady-state. The regression equation was:

$$\ln\widetilde{q}_{j}\_1985=a+b_{1}\cdot\left[\ln\left(\frac{I_{j}}{GDP_{j}}\_1985\right)-\ln\left(\delta+g+n_{j}\right)\_1985\right]+e_{j} \tag{4}$$

where j is a country index, a denotes the state of technology (a constant in the cross-section), b_1 denotes $\frac{\alpha}{(1-\alpha)}$ and $(g+\delta)$ was assumed to be 0.05 (which in my opinion is too low). They obtained an implied α (capital share) of 0.36 for a regression run on 1985 data for 22 OECD countries. However, the estimate of b_1 was insignificant and the fit was poor – $\overline{R}^2 = 0.06$. Aside from the insignificant slope estimate and the lousy fit (two big asides to be sure), this result is neoclassical-consistent. By contrast to much of the world, OECD countries enjoy relatively benign/market-supporting institutions; therefore neoclassical thinking (which implicitly assumes institutional 'neutrality') has a better chance of working in such economies than anywhere else.

Using the same regression setup for 98 non-OECD countries, MRW obtained an implied α of about 0.60 and (surprisingly, I think) a significant b_1 estimate and better fit; $\overline{R}^2 = 0.59$. The estimate $\alpha \simeq 0.6$ is a reflection of the inconsistency identified above when I imposed $\alpha = 1/3$ and solved for the saving rate differential that had to exist between rich and poor economies in order to accommodate the 30-fold plus differential in output per worker observed in data. One should conclude that equations like (4) and, hence, the 'narrow-capital' growth equations upon which they are based, are not good general models.

The endogenous saving approach of the Ramsey-Cass-Koopmans setup does not help much empirically, at least in the specific case of Cobb-Douglas production and CIES utility of consumption. Yet in my view it does yield some insight into the sources of the neoclassical model's deficiency in data. Recall from the 'Growth Theory Part 2' lecture that for CIES utility of consumption we obtained the results

$$\frac{\dot{c}}{c} = \sigma \cdot \left[f'(k) - \delta - \rho - \frac{g}{\sigma} \right]. \tag{5}$$

$$f'(k^*) = \left(\delta + \rho + \frac{g}{\sigma} \right) \tag{6}$$

For Cobb-Douglas production the above equation implies

$$\alpha k^{*\alpha - 1} = \left(\delta + \rho + \frac{g}{\sigma} \right) \tag{7}$$

$$k^* = \left[\frac{\alpha}{\left(\delta + \rho + \frac{g}{\sigma} \right)} \right]^{\frac{1}{(1-\alpha)}}. \tag{8}$$

Finding the steady-state capital stock required

$$sq^* = (\delta + g + n) \cdot k^* \tag{9}$$

which for Cobb-Douglas means

$$sk^{*\alpha} = (\delta + g + n) \cdot k^* \tag{10}$$

$$
\begin{aligned}
s &= (\delta + g + n) \cdot k^{*1-\alpha} &(11)\\
&= \frac{(\delta + g + n) \cdot \alpha}{\left(\delta + \rho + \frac{g}{\sigma}\right)}.
\end{aligned}
$$

The relative income of rich ("leading") and poor economies in this setting is therefore given by

$$
\begin{aligned}
\frac{\widetilde{q}_w^*}{\widetilde{q}_j^*} &= \frac{A_t \cdot k_w^{*\alpha}}{A_t \cdot k_j^{*\alpha}} &(12)\\
&= \left[\frac{\frac{\alpha}{\left(\delta+\rho_w+\frac{g}{\sigma_w}\right)}}{\frac{\alpha}{\left(\delta+\rho_j+\frac{g}{\sigma_j}\right)}} \right]^{\frac{\alpha}{(1-\alpha)}}\\
&= \left[\frac{\left(\delta + \rho_j + \frac{g}{\sigma_j}\right)}{\left(\delta + \rho_w + \frac{g}{\sigma_w}\right)} \right]^{\frac{\alpha}{(1-\alpha)}}.
\end{aligned}
$$

Inspection of this result shows that in order to account for a thirty-fold difference in relative income per worker (not to mention the more than 100-fold difference observed for the extremities of the international distribution of productivity) there must be huge differences in the discount rates and/or in the willingness to substitute intertemporally among people in rich and poor economies.

Consider some trial values: Set $\delta = 0.05$, $g = 0.025$, and $\alpha = 1/3$. Now assume all agents have discount rates equal to 0.03 (a typical real interest rate) and that workers in leading economies have elasticities of substitution equal to 1.0 (log utility). To generate an income ratio of 30, the elasticity of substitution among workers in poor economies would need to be 0.00026; in other words in poor countries people would have practically no willingness at all to substitute intertemporally.

Now set everyone's elasticity of substitution to 1.0 (everyone has log utility of income) and assume a standard discount rate of 0.03 among agents in rich countries. To generate the thirty-fold gap in relative income, agents in poor countries would have to have a discount rate of 94.4 per year. In other words, a unit of consumption just one year in the future would receive a utility weight barely one percent of that placed upon a unit of consumption taken this year $\left(\frac{1+0.03}{1+94.4} = 0.0108\right)$.

These simple examples reveal (what I think is) an important point: Under traditional assumptions, agents in rich economies as compared to poor must have vastly more confidence in the future – either enormously greater willingness to substitute intertemporally or dramatically lower rates of discounting future payoffs from current saving, or both. (Undoubtedly it is both.) And for these reasons rich countries build much bigger (and undoubtedly more productive) stocks of capital. Matters get less extreme, however, if we broaden capital to include human skills (as below).

1.2 *Human Capital Augmentation*

MRW (QJE, 1992) did much to revive support for the neoclassical model by explicitly incorporating human capital as a separate productive factor ("multiple" capital goods). The model is

$$Q = K^{\alpha} H^{\beta} \left(A_t \cdot N\right)^{1-\alpha-\beta} \tag{13}$$

with $0 < \alpha, \beta < 1$, $0 < [1 - (\alpha + \beta)] < 1$, and again $\dot{A} = gA$ and $\dot{N} = nN$. In intensive form:

$$q = k^{\alpha} h^{\beta}. \tag{14}$$

And in per worker form[2]

$$\widetilde{q} = A_t \cdot k^{\alpha} h^{\beta}. \tag{15}$$

Physical capital accumulates by

$$\dot{K} = s_k Q - \delta K \tag{16}$$

$$\dot{k} = s_k q - (\delta + n + g) \cdot k \tag{17}$$

and human capital accumulates by[3]

$$\dot{H} = s_h Q - \delta H \tag{18}$$

[2] Alternatively, $\widetilde{q} = A_t^{(1-\alpha-\beta)} \cdot \widetilde{k}^{\alpha} \widetilde{h}^{\beta}$.

[3] Recall $\frac{d \ln k}{dt} \equiv \frac{1}{k} \frac{dk}{dt} \equiv \frac{\dot{k}}{k}$. So $\frac{\dot{k}}{k} = \left(\frac{d \ln K}{dt} - \frac{d \ln A}{dt} - \frac{d \ln N}{dt}\right) = \left(\frac{\dot{K}}{K} - \frac{\dot{A}}{A} - \frac{\dot{N}}{N}\right)$, and $\frac{\dot{k}}{k} = \left[\left(s \frac{Q}{K} - \delta\right) - (n + g)\right]$, $\dot{k} = k \frac{\dot{k}}{k} = k \cdot \left[\left(s \frac{Q}{K} - \delta\right) - (n + g)\right] = sq - (\delta + n + g)k$. The same naturally holds for $\dot{h}$.

$$\dot{h} = s_h q - (\delta + n + g) \cdot h. \tag{19}$$

The steady-states therefore are[4]

$$k^* = \left[\frac{s_k^{(1-\beta)} s_h^{\beta}}{(\delta + n + g)} \right]^{\frac{1}{(1-\alpha-\beta)}} \tag{20}$$

$$h^* = \left[\frac{s_k^{\alpha} s_h^{(1-\alpha)}}{(\delta + n + g)} \right]^{\frac{1}{(1-\alpha-\beta)}}. \tag{21}$$

For the real object of interest we obtain

$$\begin{aligned} \widetilde{q}^* &= A_t \cdot k^{*\alpha} h^{*\beta} \qquad (22) \\ &= A_t \cdot \left[\frac{s_k^{(1-\beta)} s_h^{\beta}}{(\delta + n + g)} \right]^{\frac{\alpha}{(1-\alpha-\beta)}} \left[\frac{s_k^{\alpha} s_h^{(1-\alpha)}}{(\delta + n + g)} \right]^{\frac{\beta}{(1-\alpha-\beta)}}. \end{aligned}$$

Hence, relative incomes under this human capital-augmented neoclassical regime are[5]

$$\frac{\widetilde{q}_w^*}{\widetilde{q}_j^*} = \left(\frac{s_{kw}}{s_{kj}} \right)^{\frac{\alpha}{(1-\alpha-\beta)}} \cdot \left(\frac{s_{hw}}{s_{hj}} \right)^{\frac{\beta}{(1-\alpha-\beta)}} \cdot \left[\frac{(\delta + g + n_j)}{(\delta + g + n_w)} \right]^{\frac{\alpha+\beta}{(1-\alpha-\beta)}}. \tag{23}$$

One can readily see that this model has much better chance of plausibly accounting for observed relative incomes. Assume, as we did with the bare bones Cobb-Douglas setup, that the last ratio in brackets on the right-side is no greater than 1.7. The stock of human capital – to be thought of as the market value of the labor force's acquired skills – may well exceed that of physical capital, at least in developed economies. Moreover, returns to human capital probably exceed returns to physical capital. (See the sources cited in Hibbs, Kyklos 2001). Setting $\alpha = \beta = 1/3$, we find that (23) implies

$$\frac{\widetilde{q}_w^*}{\widetilde{q}_j^*} = \left(\frac{s_{kw}}{s_{kj}} \right) \cdot \left(\frac{s_{hw}}{s_{hj}} \right) \cdot 2.9. \tag{24}$$

[4] I suppose it's obvious, but see the Appendix for a (very boring) derivation.

[5] I have terribly finite cerebral RAM, and so all the exponents start to confuse me. If you do to, it may be easier to take logs, work with linear functions and then exponentiate to obtain the income relatives shown.

Therefore, for $\frac{\tilde{q}_w^*}{\tilde{q}_j^*} = 30$ we need $\left(\frac{s_{kw}}{s_{kj}}\right) \cdot \left(\frac{s_{hw}}{s_{hj}}\right) = 10.4$, which is easily consistent with international data. Measured by average formal years of schooling of the workforce, $\left(\frac{s_{hw}}{s_{hj}}\right)$ alone can get up to 8 or so.

Not surprisingly, then, fitting a regression for log output per working age person based on eq.(22), MRW obtain higher coefficient significance levels and better overall fits in their 1985 cross-section than they did with the bare bones Cobb-Douglas equation. Outside the OECD, the implied values of α and β were around 0.3. Within the OECD, the sum of capital returns $(\alpha + \beta)$ was comparable, but the estimated return to human capital (measured extremely crudely and inappropriately, as a flow not a stock, by secondary school enrollment data) was around two and half times bigger than the return to physical capital. This is quite plausible given that the quality of a typical year of schooling is probably higher in rich countries than in poor.

The conclusion that follows from the above is that a rather simple human capital augmented Cobb-Douglas model – a model in which the concept of capital is broadened to include the stock of productive skills embodied in raw labor or, essentially equivalently, the concept of saving is broadened to include the implicit costs of acquiring skills in the form of both time and direct expenditure – fares pretty well when confronted with international data.

We are left with the question, however, of why we observe such big differences in human (as well as physical) capital formation in the first instance. The answer I am convinced lies with international differences in institutional arrangements that affect the willingness and capacity of agents to assemble capital (as well as with the productivity of the capital that winds up being deployed in production). I have more to say about this in Hibbs (2001), which directs you to lots of literature. Important things are also said, and creative original empirical research is reported, in Hall and Jones (1999).

2 Conditional Convergence Relations in Discrete Time ('Barro Regressions')

The continuous time relations derived in previous lectures showed that when k is not too far from k^*[6]

$$\frac{d \ln q(t)}{dt} \simeq \beta \cdot (\ln q^* - \ln q(t)), \tag{25}$$

[6] See the 'Growth Theory Part 1' lecture.

implying

$$\ln q(t) - \ln q^* = e^{-\beta t}\left[\ln q(0) - \ln q^*\right] \tag{26}$$

where recall β denotes $(\delta + g + n) \cdot (1 - \alpha_{k^*})$ .In discrete time (as in the Overlapping Generations setup) we could derive the corresponding partial adjustment equation

$$\ln q_t - \ln q_{t-1} = (1 - \lambda) \cdot [\ln q^* - \ln q_{t-1}] \tag{27}$$

where $0 < \lambda < 1$, $\lambda \simeq e^{-\beta}$ is the parameter determining the proportion of the gap between $\ln q$ and $\ln q^*$ that is closed each period.[7] This is true of any neoclassical production function, not just the Cobb-Douglas, as we have already seen. The partial adjustment model implies

$$\ln q_t (1 - \lambda L) = (1 - \lambda) \cdot \ln q^*. \tag{28}$$

Multiplying through by $\left(1 + \lambda L + \lambda^2 L^2 + ... + \lambda^{t-1} L^{t-1}\right)$ gives

$$\ln q_t - \lambda^t \cdot \ln q_0 = (1 - \lambda) \frac{\left(1 - \lambda^t\right) L^t}{(1 - \lambda)} \cdot \ln q^* \tag{29}$$

$$\ln q_t = \lambda^t \cdot \ln q_0 + \left(1 - \lambda^t\right) \cdot \ln q^* \tag{30}$$

where again note that $\left(1 - \lambda^t\right) \simeq \left(1 - e^{-\beta t}\right)$.

Now let's get an equation for output per worker. Recall $\ln q = A^{-1} \cdot \ln \widetilde{q}$, so eq.(30) can be written

$$\ln\left(A_t^{-1} \cdot \widetilde{q}_t\right) = \lambda^t \cdot \ln\left(A_0^{-1} \cdot \widetilde{q}_0\right) + \left(1 - \lambda^t\right) \cdot \ln q^* \tag{31}$$

$$\begin{aligned} \ln \widetilde{q}_t &= -\ln A_t^{-1} + \lambda^t \ln A_0^{-1} + \lambda^t \cdot \ln \widetilde{q}_0 + \left(1 - \lambda^t\right) \cdot \ln q^* \\ &= \ln A_t - \lambda^t \ln A_0 + \lambda^t \cdot \ln \widetilde{q}_0 + \left(1 - \lambda^t\right) \cdot \ln q^*. \end{aligned} \tag{32}$$

Hence from period 0 to period t the cumulative growth of output in country j is

$$\ln \widetilde{q}_{jt} - \ln \widetilde{q}_{j0} = \ln A_t - \lambda^t \ln A_0 - \left(1 - \lambda^t\right) \cdot \ln \widetilde{q}_{j0} + \left(1 - \lambda^t\right) \cdot \ln q^*_{j(t)}. \tag{33}$$

[7]Notice that when the model is expressed as $gap_t \equiv [\ln q_t - \ln q^*] = \lambda^t \cdot [\ln q_0 - \ln q^*]$, the adjustment coefficient $(1 - \lambda)$ has the same interpretation as β in the continuous time model: $\frac{1}{gap}\frac{d[gap]}{dt} \equiv \frac{d \ln gap}{dt} = \ln \lambda$. Hence $\ln q_t$ converges to its steady-state value $\ln q^*$ at a rate per period equal to $-(1 - \lambda) \simeq \ln \lambda \simeq -\beta$.

As we saw in the Neoclassical Growth Theory lecture, the implication is that the growth rate of output per worker (here expressed cumulatively from period-0 to period-t) is higher the greater the gap between initial condition output per worker and the steady-state per effective worker, with convergence rate given by $\left(1-\lambda^{t}\right)$.[8] But notice that I have indexed the level of steady-state output per effective worker for j and t. Why? Estimating cross-section or panel regressions assuming a common $\ln q^{*}$ fares terribly in data. (See the empirics in Barro and Sala-i-Martin.) In fact, cumulative postwar growth rates in relation to initial conditions for a broad international cross-section of countries (which implicitly assumes that q^{*} is the same for all countries) indicates that there is actually a slightly positive relation between the two.

Yet nothing in the neoclassical model assumes that steady-states should be constant across time or space (economies). As you know well, steady-states depend upon saving rates, stocks of capital (human and physical) which may vary through time and space, along with the parameters of the model environment, which in this line of research are taken to be common. Hence in the 'Barro regression' approach, researchers attempt to find the 'ultimate' or 'fundamental' sources of variation in $\ln q^{*}$. The regression setups are almost always linear and, hence, are based on an implicit specification

$$\ln q_{j(t)}^{*}=a+X_{j(t)}b \tag{34}$$

where X is a matrix of possible 'fundamental' determinates of potential income per worker in economy j at time t. This yields test equations in the form

$$\ln\widetilde{q}_{jt}-\ln\widetilde{q}_{j0}=\ln A_{t}-\lambda^{t}\ln A_{0}-\left(1-\lambda^{t}\right)\cdot\ln\widetilde{q}_{j0}+\left(1-\lambda^{t}\right)\cdot\left(a+X_{j(t)}b\right). \tag{35}$$

All parameters in this equation are obviously not identified. In a pure cross-section of cumulative growth rates (which could of course span just 1 period), the variable term(s) $\ln A_{t}-\lambda^{t}\ln A_{0}$ are constant, and the parametric term $\left(1-\lambda^{t}\right)$ is also a constant. A feasible OLS regression is therefore

$$\ln\widetilde{q}_{jt}-\ln\widetilde{q}_{j0}=C_{0}+C_{1}\ln\widetilde{q}_{j0}+C_{2}X_{j(t)} \tag{36}$$

where C_{0} would represent $\left[\ln A_{t}-\lambda^{t}\ln A_{0}+\left(1-\lambda^{t}\right)\cdot a\right]$, $C_{1}=\left(1-\lambda^{t}\right)$, and $C_{2}=\left(1-\lambda^{t}\right)\cdot b$. It should be clear how the convergence rate could be deduced from C_{1}. Once you have λ from C_{1}, you can infer b from C_{2}. In a panel or a single

[8] We could of course express all the $q's$ as output per effective worker and get rid of the $A's$, but this would not be suited to empirical work.

economy time-series regression one would have to commit to a specification of $\ln A_t$ (the most typical being a linear trend), or alternatively (and probably more desirably) by specifying the model with period effects to pick up the state of A at each t. Another possibility is to deviate $\ln \widetilde{q}_{jt}$ from the the world technology leader's productivity at each period, on the assumption that the state of technology is thereby netted out.

Research based on equations in the form of (36) have investigated the impact of many dozens of "X" variables in many thousands of combinations. By my reading of the literature (and there are now a great number of papers), institutional variables calibrating the security of property rights consistently turn out to be the most robust fundamental determinants of output growth. The research of Barro, Barro and Sala-i-Martin and many others frequently estimate a convergence rate in the vicinity of 2 percent per annum in such regression regimes ("$\beta-convergence$").

Note, however, that if the X 'driving variables' are relatively stable in the data (institutional characteristics often are quite stable through time), then the growth rates observed may already be driven mainly by the (assumed common) rate of technological progress. In this case, which I believe is quite prevalent, the investigator would fail to pick up the full effects of 'outside' fundamental determinants of growth and prosperity; such effects would already be embedded in the level of output, and not show up in its rate of growth. Moreover, the parameters that are assumed by this setup to be common to all economies (notably the parameter(s) of production, which determine the efficiency with which inputs are transformed to output value added) may themselves in fact be variable, due to the effects of institutional variables.[9] This has led many researchers (Hall and Jones 1999 is an most important example) to move to reduced form cross-sectional empirical equations of the form:

$$\ln q_{j(t)} = a + X_{j(t)} b. \tag{37}$$

Note that such test equations are wholly detached from the last remnant of neoclassical theory– the required negative "convergence parameter" on the initial condition of an output growth equation.[10]

Appendix on the Steady-States of the Mankiw et al. (1992) Setup

[9] Obvious examples can be found in the planned economies of the former Soviet bloc, where in some sectors the shadow market value of output produced was probably worth less than the shadow market values of the labor, capital and raw matterial inputs. Plausible assertions of a similar sort have been made about 'wasted capital' under 'crony capitalism' in some East Asian countries.

[10] And even this connection to theory is weak. Note too how the convergence parameter relates to the rather large time-series literature on the stochastic properties of various macroeconomic variables, especially the issue of whether or not output has a 'unit root'.

As you know, you must solve the equations of motion $\dot{k} = 0$ and $\dot{h} = 0$ simultaneously.

The long (more intuitive?) route uses successive elimination (the more elegant and faster approach, if you have your matrix algebra in firm grasp, would solve the simultaneous equations problem using standard matrix solution concepts). It goes like this:

(i) $s_k q^* = (\delta + n + g)\, k^*$, (ii) $s_k k^{*\alpha} h^{*\beta} = (\delta + n + g)\, k^*$,

(iii) $k^{*\alpha - 1} = (\delta + n + g)\, s_k^{-1} h^{*-\beta}$ (iv) $k^* = \left[(\delta + n + g)^{-1} s_k h^{*\beta}\right]^{\frac{1}{(1-\alpha)}}$.

Now the same for h:

(v) $s_h q^* = (\delta + n + g)\, h^*$, (vi) $s_h k^{*\alpha} h^{*\beta} = (\delta + n + g)\, h$,

(vii) $h^{*\beta - 1} = (\delta + n + g)\, s_h^{-1} k^{*-\alpha}$ (viii) $h^* = \left[(\delta + n + g)^{-1} s_h k^{*\alpha}\right]^{\frac{1}{(1-\beta)}}$.

Now substitute (viii) in (iv). But all these exponents are starting to confuse me. Maybe you too. So let's take logs first and then make the substitution. After logging we obtain for (iv):

(ix) $\ln k^* = \frac{1}{(1-\alpha)} \cdot [-\ln(\delta + n + g) + \ln s_k + \beta \ln h^*]$. After making the substitution for $\ln h^*$ you get

(x) $\ln k^* = \frac{1}{(1-\alpha)} \cdot \begin{bmatrix} -\ln(\delta + n + g) + \ln s_k \\ +\frac{\beta}{(1-\beta)}[-\ln(\delta + n + g) + \ln s_h + \ln \alpha k^*] \end{bmatrix}$. Now combine the $\ln(\delta + n + g)$ terms and get $\ln k^*$ to the left-side,

(xi) $\ln k^* \left[1 - \frac{\alpha\beta}{(1-\alpha)\cdot(1-\beta)}\right] = \frac{1}{(1-\alpha)} \cdot \begin{bmatrix} -\frac{1}{(1-\beta)} \ln(\delta + n + g) \\ +\ln s_k + \frac{\beta}{(1-\beta)} \ln s_h \end{bmatrix}$. Next convince yourself that $\left[1 - \frac{\alpha\beta}{(1-\alpha)\cdot(1-\beta)}\right] = \frac{(1-\alpha-\beta)}{(1-\alpha)\cdot(1-\beta)}$ (it does), then invert it and multiply through. You should get

(xii) $\ln k^* = \begin{bmatrix} -\frac{1}{(1-\alpha-\beta)} \ln(\delta + n + g) \\ +\frac{1}{(1-\alpha-\beta)}(1-\beta)\ln s_k + \frac{1}{(1-\alpha-\beta)}\beta \ln s_h \end{bmatrix}$. Now exponentiate and viola' you get the result in the main lecture:

(xiii) $k^* = \left[\frac{s_k^{(1-\beta)} s_h^{\beta}}{(\delta+n+g)}\right]^{\frac{1}{(1-\alpha-\beta)}}$. Repeat steps (ix) to (xiii) for h^* to retrieve the main lecture result. It's too boring so I leave it to you.

Lecture Reference Notes

A lot of what I have done here is lifted from Hibbs (Kyklos, 2001). The Barro and Sala-i-Martin chapters assigned for this topic report lots of good empirics using the "Barro regression", conditional $\beta-$convergence approach. You should be familiar with the considerable evidence favoring the neoclassical model supplied by this line of research. Hall and Jones (1999) is really worth reading. It helped stimulate a flood of papers on institutional sources of international variation in prosperity. A truly innovative study (if I don't say so myself), though perhaps a bit 'over the top' in some respects, that focuses on very long-run development issues (13,000 years!) is Hibbs and Olsson (PNAS, 2004).