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Neoclassical Growth Theory with Exogenous Saving

(Solow-Swan)

1 Stylized Facts

- $\dot{(Q/N)} > 0$, steady secular growth
- $\dot{(K/N)} > 0$, physical capital to labor ratio grows
- $r = F'K \equiv \frac{\partial Q}{\partial K}$ seems to fall with development, that is, the marginal product of capital is lower in mature industrial/post-industrial economies than in less developed ones that are growing (an important qualification)
- $\frac{K}{Q}$, capital to output ratio is relatively constant in mature economies
- $\frac{wN}{Q}$, $\frac{rK}{Q}$,factor shares are relatively constant
- $\sigma_i^2\dot{(Q/N)}$, $\quad \sigma_i^2\,(Q/N)$, (where i indexes national economies) are too "big" to be consistent with the standard neoclassical model
- Politics, policies and institutions are increasingly recognized as the key to explaining the broad international and intertemporal patterns in growth and development

2 The Basic Neoclassical Model

$$Q = F(A, K, N) \tag{1}$$

where A is the exogenously given, commonly available state of technology, K is proprietary (rivalrous) capital input, and N is proprietary labor input. K and N are homogenous in productive quality.

Properties:

- $F(\cdot)$ is twice differentiable and homogeneous of degree 1 ($h = 1$) in N and K inputs:

$$\begin{aligned} K \cdot F'(K) + N \cdot F'N &= h \cdot F(A, K, N), \qquad h = 1 \\ F(A,\ \mu \cdot K,\ \mu \cdot N) &= \mu \cdot F(A, K, N), \qquad \textit{all } \mu > 0 \end{aligned}$$

- For $K > 0$, $N > 0$,the function $F(\cdot)$ has positive and diminishing marginal products:

$$F'(K) > 0,\ F'(N) > 0, \qquad F''(K) < 0,\ F''(N) < 0$$

- Inada conditions on input extremities:

$$\begin{aligned} K \overset{\lim}{\rightarrow} 0F'(K) &= \infty, \qquad N \overset{\lim}{\rightarrow} 0F'(N) = \infty \\ K \overset{\lim}{\rightarrow} \infty F'(K) &= 0, \qquad (N) \overset{\lim}{\rightarrow} \infty F'(N) = 0 \end{aligned}$$

2.1 The Intensive Form

To find steady-states with growing technology, we must assume A is embodied in N (or is a disembodied productivity 'shifter', as in the Cobb-Douglas function[1].

[1] See Barro and Sala-i-Martin (2004), appendix 1.5.3 for a proof. Note that the famous Cobb-Douglas production function

$$Q = AK^{\alpha}N^{1-\alpha}$$

satisfies the labor-augmenting technological progress condition, as it could just as well be written,

$$\begin{aligned} Q &= K^{\alpha}(AN)^{1-\alpha} \\ &= \widetilde{A}K^{\alpha}N^{1-\alpha} \end{aligned}$$

with $\widetilde{A} = A^{1-\alpha}$, which just re-scales A.

So I express the model as

$$Q = F(K, A \cdot N) \tag{2}$$

The degree 1 homogeneity of $F(\cdot)$ allows the model to be written:

$$\begin{aligned} Q &= N \cdot F\left(\frac{K}{N}, 1\right) \\ &= N \cdot F\left(\frac{K}{N}, 1\right) \\ &= N \cdot f(k) \end{aligned} \tag{3}$$

Hence the model in "intensive" (per "effective" unit of labor input) form is:

$$q = f(k) \tag{4}$$

where $q \equiv \frac{Q}{AN}$, $k \equiv \frac{K}{AN}$,and $f \equiv F(k, 1)$.

Note that the intensive form representation has the same properties as the model in basic form, concavity and Inada conditions at the extremities:

$$\begin{aligned} f'(k) &> 0, \qquad f''(k) < 0 \\ f'(0) &= \infty, \qquad f'(\infty) = 0 \end{aligned}$$

For example, $f'(k) = F'(K)$:

$$\begin{aligned} F'(K) &\equiv \frac{\partial F(K, AN)}{\partial K} = \frac{\partial \left[AN \cdot F\left(\frac{K}{AN}, 1\right)\right]}{\partial K} \\ &= \frac{\partial [AN \cdot f(k)]}{\partial k} \cdot \frac{\partial k}{\partial K} = \frac{\partial [AN \cdot f(k)]}{\partial k} \cdot \frac{\partial \left[K \cdot (AN)^{-1}\right]}{\partial K} \\ &= AN \cdot f'(k) \cdot (AN)^{-1} = f'(k). \end{aligned} \tag{5}$$

[*Graph* $q = f(k)$]

2.2 Input Laws of Motion

2.3 *Labor*

$$\frac{\dot{N}}{N} = n$$

$$N(t) = N(0) \cdot e^{nt}, \qquad N(0) = 1.$$

2.4 *Technology/Knowledge/Productivity*

$$\frac{\dot{A}}{A} = g$$

$$A(t) = A(0) \cdot e^{gt}, \qquad A(0) = 1.$$

2.5 The Fundamental Dynamic Equation (capital)

The economy is closed, so:

$$Q = C + I \tag{6}$$

$$S = I = (Q - C). \tag{7}$$

The equation of motion for the aggregate stock of physical capital is:

$$\begin{aligned} \dot{K} &= I - \delta K && (8) \\ \dot{K} &= sQ - \delta K && (9) \end{aligned}$$

where $s \equiv \frac{S}{Q} \equiv \frac{I}{Q}$ and $0 < \delta < 1$.

Now find $\dot{k} \equiv \frac{d\left[K \cdot (AN)^{-1}\right]}{dt}$:

$$\begin{aligned} \dot{k} &= \frac{\partial\left[K \cdot (AN)^{-1}\right]}{\partial K} \cdot \frac{dK}{dt} + \frac{\partial\left[K \cdot (AN)^{-1}\right]}{\partial AN} \cdot \frac{d(AN)}{dt} && (10) \\ &= (AN)^{-1} \dot{K} - K (AN)^{-2} \cdot \dot{(AN)} \\ &= (AN)^{-1} (sQ - \delta K) - k \frac{\dot{(AN)}}{AN} \\ &= sq - \delta k - k \cdot (g + n) \\ &= sq - (\delta + g + n) k \end{aligned}$$

$$\dot{k} = sf(k) - (\delta + g + n) k. \tag{11}$$

The curvature of production, along with the Inada conditions, insure that there is a unique stead-state level of capital intensity per worker, k^*, at which $\dot{k} = 0$ satisfying $sq = (\delta + g + n) k$.

[*Graphs*: $sf(k)$, $(\delta + g + n)k$, k, k^*, phase diagram $\dot{k}$ *wrt* k.]

2.6 Output Growth Per Effective Worker

Given $q = f(k)$, the growth rate of output per effective worker is

$$\frac{\dot{q}}{q} \equiv \frac{1}{q}\frac{dq}{dt} = \frac{1}{q}\frac{\partial f(k)}{\partial k} \cdot \frac{dk}{dt} = \frac{1}{q} f'(k)\dot{k} \tag{12}$$

which upon multiplication of the right-side by $\frac{k}{k}$ yields

$$\frac{\dot{q}}{q} = \frac{f'(k) \cdot k}{q}\frac{\dot{k}}{k} = \alpha_k \frac{\dot{k}}{k} \tag{13}$$

where α_k denotes capital's share of output, $\frac{f'(k) \cdot k}{q}$, if capital commands its marginal product at every instant.[2]

At $k = k^*$ and, hence, at $\frac{\dot{k}}{k} = 0$, the growth rate of output per effective worker is therefore zero:

$$\frac{\dot{q}}{q}\,(k = k^*) = 0. \tag{14}$$

We calibrate "prosperity", however, in terms of output per worker. Define the per worker quantities:

$$\tilde{q} \equiv \frac{Q}{N}, \qquad \tilde{k} \equiv \frac{K}{N}.$$

By definition

$$\frac{\dot{k}}{k} \equiv \frac{1}{k}\frac{dk}{dt} \equiv \frac{d \ln k}{dt} \equiv \frac{d \ln \left[K \cdot (AN)^{-1}\right]}{dt} \equiv \left(\frac{\dot{K}}{K} - \frac{\dot{A}}{A} - \frac{\dot{N}}{N}\right) \tag{15}$$

$$= \frac{\dot{\tilde{k}}}{\tilde{k}} - g \tag{16}$$

[2]Note that since we know that $f'(k) = F'(K)$, the capital share is the same in the generic form of the model as in the intensive form:
$\alpha_k = \frac{f'(k) \cdot k}{q} = \frac{F'(K) \cdot (K/N)}{(Q/N)} = \frac{F'(K) \cdot K}{Q} = \alpha_K.$

and

$$\frac{\dot{q}}{q} \equiv \frac{1}{q}\frac{dq}{dt} \equiv \frac{d\ln q}{dt} \equiv \frac{d\ln\left[Q\cdot(AN)^{-1}\right]}{dt} \equiv \left(\frac{\dot{Q}}{Q} - \frac{\dot{A}}{A} - \frac{\dot{N}}{N}\right) \tag{17}$$

$$= \frac{\dot{\tilde{q}}}{\tilde{q}} - g \tag{18}$$

where recall g is the growth rate of $A(t)$.

Given

$$\frac{\dot{q}}{q} = \alpha_k \frac{\dot{k}}{k}$$

we know

$$\frac{\dot{\tilde{q}}}{\tilde{q}} - g = \alpha_k \left(\frac{\dot{\tilde{k}}}{\tilde{k}} - g\right) \tag{19}$$

$$\frac{\dot{\tilde{q}}}{\tilde{q}} = \alpha_k \frac{\dot{\tilde{k}}}{\tilde{k}} + (1-\alpha_k)g.$$

And since consumption per worker is

$$\tilde{c} = (1-s)\,\tilde{q} \tag{20}$$

it grows at the same rate as output for any exogenously give savings rate:

$$\frac{\dot{\tilde{c}}}{\tilde{c}} = \frac{1}{\tilde{c}}(1-s)\,\dot{\tilde{q}}$$

$$= \frac{\dot{\tilde{q}}}{\tilde{q}} \tag{21}$$

At steady-state $k = k^*$, $\frac{\dot{k}}{k} = \frac{\dot{q}}{q} = 0$, the implication is that output and capital and consumption per unit labor input grow at the rate of exogenous technological progress, g, and are therefore independent of parameters of production and rates of saving, depreciation and labor force growth (*and anything else, including politics, policies and institutional arrangements*):

$$\frac{\dot{\tilde{q}}}{\tilde{q}}|_{(k=k^*)} = \frac{\dot{\tilde{k}}}{\tilde{k}}|_{(k=k^*)} = \frac{\dot{\tilde{c}}}{\tilde{c}}|_{(k=k^*)} = g. \tag{22}$$

3 The Level of Output Per Worker at Steady-State

At steady-state, all we can say about the (technology-conditional) convergent level of output per worker is

$$\begin{aligned} \tilde{q}^* &= A(t) \cdot q^* \\ &= A(t) \cdot f(k^*) \end{aligned}$$

where k^* will be some function of the given parameters α, δ, g and n. To say more we need to specify $F(\cdot)$ – the functional form of production.

In the Cobb-Douglas case, the convergent level of output per worker would be

$$\tilde{q}^*|_{A(t)} = A(t) k^{*\alpha}, \qquad \alpha \in (0, 1). \tag{23}$$

k^* it is readily derived from the first-order differential equation for capital, evaluated at $\dot{k} = 0$. With Cobb-Douglas production we have

$$\dot{k} = s k^{\alpha} - (\delta + g + n) k \tag{24}$$

which at $\dot{k} = 0$ implies

$$s k^{*\alpha} = (\delta + g + n) k^*. \tag{25}$$

Solving for k^* gives steady-state capital per effective worker as

$$k^* = \left[\frac{s}{(\delta + g + n)} \right]^{\frac{1}{(1-\alpha)}}. \tag{26}$$

Convergent output worker is then

$$\begin{aligned} \tilde{q}^*|_{A(t)} &= A(t) \cdot q^* = A(t) \cdot k^{*\alpha} \qquad (27) \\ &= A(t) \cdot \left[\frac{s}{(\delta + g + n)} \right]^{\frac{\alpha}{(1-\alpha)}} \end{aligned}$$

We shall revisit this result when considering the relative incomes of economies (the wealth and poverty of nations).

4 Effects of Changes in the Rate of Saving

We aim to learn how a change in the exogenous rate of saving affects the steady-state values of capital, output and consumption per worker.

Steady-State Capital per Effective Worker (k^*)*:*

We already know that when $k = k^*$, $\dot{k} = 0$. So

$$sf(k^*) = (\delta + g + n)\, k^*. \tag{28}$$

Therefore we can find $\frac{dk^*}{ds}$by implicit differentiation:

$$\frac{d\left[sf(k^*)\right]}{ds} = \frac{d\left[(\delta + g + n)\, k^*\right]}{ds} \tag{29}$$

$$f(k^*) + s \cdot f'(k^*) \cdot \frac{dk^*}{ds} = (\delta + g + n) \cdot \frac{dk^*}{ds} \tag{30}$$

$$f(k^*) = \left[(\delta + g + n) - s \cdot f'(k^*)\right] \cdot \frac{dk^*}{ds}. \tag{31}$$

Solving for $\frac{dk^*}{ds}$ gives

$$\frac{dk^*}{ds} = \frac{f(k^*)}{\left[(\delta + g + n) - s \cdot f'(k^*)\right]} > 0. \tag{32}$$

The response of the steady-state capital stock is unambiguously positive because $s \cdot f'(k^*)$ is always less than $(\delta + g + n)$, though it need not be less than $f'(k^*)$, given the possibility of 'under saving' in the Solow model. [*See Graph on the "golden rule" introduced ahead in the lecture*]

Steady-State Output per Effective Worker (q^*)*:*

$$\frac{dq^*}{ds} \equiv \frac{df(k^*)}{ds}$$

$$\frac{dq^*}{ds} = f'(k^*)\frac{dk^*}{ds} > 0. \tag{33}$$

The response of steady-state output is unambiguously positive since both $f'(k^*)$ and $\frac{dk^*}{ds}$ are positive.

Steady-State Consumption per Effective Worker (c^*)*:*

Recall $C = (Q - I)$, and $s \equiv \frac{I}{Q} = \frac{S}{Q}$. Let $c \equiv \frac{C}{AN} = (\frac{Q}{AN} - \frac{I}{AN})$. Hence,

$$c = (q - sq)$$
$$c = (1 - s) \cdot q$$
$$c = (1 - s) \cdot f(k)$$

As noted repeatedly, at steady-state $sf(k^*) = (\delta + g + n)\,k^*$, which implies that at steady-state any given saving rate may be expressed:

$$s_{(k^*)} = \frac{(\delta + g + n)\,k^*}{f(k^*)}. \tag{34}$$

Therefore steady-state consumption per effective worker is

$$c^* = (1 - s) \cdot f(\ k^*)$$
$$c^* = \left[(1 - \tfrac{(\delta+g+n)k^*}{f(k^*)})\right] \cdot f(\ k^*)$$
$$c^* = \left[f(k^*) - (\delta + g + n)\,k^*\right].$$

The response of c^* to a change in s is then

$$\begin{aligned} \frac{dc^*}{ds} &= f'(k^*) \cdot \frac{dk^*}{ds} - (\delta + g + n) \cdot \frac{dk^*}{ds} \\ &= \left[f'(k^*) - (\delta + g + n)\right] \cdot \frac{dk^*}{ds} \gtreqless 0. \end{aligned} \tag{35}$$

$\frac{dk^*}{ds}$is unambiguously positive, so the sign of the response of c^* to a change in s depends on whether the marginal product of capital at the new steady-state capital stock that is generated by a change in s is greater or less than the slope of the break-even investment schedule, $(\delta + g + n)$. The sign of $\frac{dc^*}{ds}$is therefore uncertain.

We can say, however, that c^* is maximized at a saving rate generating a steady-state capital stock, k^*, such that $f'(k^*) = (\delta + g + n)$, which is known as the *"golden rule"* (of capital accumulation) saving rate.[3] Note that in the Solow-Swan neoclassical model saving behavior is given exogenously, so there is no endogenous/model-driven reason why s would happen take a value that maximizes c^*.[*Graphs* to illustrate]

[3] This result appears to have been discovered independently and more or less simultaneoulsy (as often happens in science) by Phelps, *AER* (1961), Swan in Berrill ed. *Economic Development* (1964), Desrousseaux, *Annales des Mines* (1961), Joan Robinson, *RES* (1962) and von Weizsäcker, *Wachstum, Zins und Optimale Investitionsquote* (1962), and probably others.

4.1 The Elasticity of q^* with Respect to the Saving Rate

Some insight is gained by expressing the response of q^* to s as an elasticity:

$$\frac{dq^*}{ds} \cdot \frac{s}{q^*}.$$

We already know that

$$\begin{aligned} \frac{dq^*}{ds} &= f'(k^*) \frac{dk^*}{ds} \\ &= f'(k^*) \cdot \left[\frac{f(k^*)}{[(\delta + g + n) - s \cdot f'(k^*)]} \right] > 0. \end{aligned}$$

Hence,

$$\frac{s}{q^*} \cdot \frac{dq^*}{ds} = \frac{s}{f(k^*)} \cdot f'(k^*) \cdot \left[\frac{f(k^*)}{((\delta + g + n) - s \cdot f'(k^*))} \right] \tag{36}$$

which after substitution for $s_{(k^*)} = \frac{(\delta + g + n) k^*}{f(k^*)}$ yields

$$\frac{s}{q^*} \cdot \frac{dq^*}{ds} = \frac{(\delta + g + n)\, k^*}{f(k^*)} \cdot \frac{1}{f(k^*)} \cdot f'(k^*) \cdot \left[\frac{f(k^*)}{\left((\delta + g + n) - \frac{(\delta + g + n) k^*}{f(k^*)} \cdot f'(k^*) \right)} \right]. \tag{37}$$

Recall that the capital share of income is $\alpha_k = \frac{f'(k) \cdot k}{f(k)}$.[4] Therefore the output elasticity of saving may be expressed

$$\begin{aligned} \frac{s}{q^*} \cdot \frac{dq^*}{ds} &= \frac{(\delta + g + n) \cdot \alpha_{k^*}}{[(\delta + g + n) - (\delta + g + n)\, \alpha_{k^*}]} \\ &= \frac{\alpha_{k^*}}{(1 - \alpha_{k^*})}. \end{aligned} \tag{38}$$

The proportional response of output per effective worker to a proportional change in the saving rate is then

$$\frac{dq^*}{q^*} = \frac{\alpha_{k^*}}{(1 - \alpha_{k^*})} \cdot \left(\frac{ds}{s} \right). \tag{39}$$

It follows, for example, that if the physical capital share of income takes a "traditional" value of around 1/3, the output elasticity is around 1/2. Doubling

[4] As before, note $\alpha_k \equiv \frac{f'(k) \cdot k}{q} = \alpha_{\tilde{k}} \equiv \frac{f'(\tilde{k}) \cdot \tilde{k}}{\tilde{q}} = \alpha_K \equiv \frac{f'(K) \cdot K}{Q}$.

the savings rate for physical capital accumulation will yield a permanent increase in output per effective worker by half as much.

The concept of "capital," and the effect of capital's income share on the response of output to saving rates, are important to evaluation of the neoclassical model's consistency with patterns in relative incomes observed through time and space. I shall revisit this point in subsequent lectures.

Finally, since $\tilde{q}^* = A(t) \cdot q^*$, the elasticity of output per worker to a change in the saving rate is of course identical to elasticity of output per effective worker derived above, because technological progress is exogenous and does not respond to investment, i.e.$\frac{dA(t)}{ds} = 0$:

$$\begin{aligned} \frac{s}{\tilde{q}^*} \cdot \frac{d\tilde{q}^*}{ds} &\equiv \frac{s}{A(t) \cdot q^*} \cdot \frac{d\left[A(t) \cdot q^*\right]}{ds} \\ &= \frac{s}{A(t) \cdot q^*} \cdot \left\{ A(t) \cdot \frac{dq^*}{ds} + q^* \cdot \frac{dA(t)}{ds} \right\} \\ &= \frac{s}{A(t) \cdot q^*} \cdot \left\{ A(t) \cdot \frac{dq^*}{ds} + 0 \right\} \\ &= \frac{s}{q^*} \cdot \frac{dq^*}{ds}. \end{aligned} \tag{40}$$

5 Transitional Dynamics

We already know that on the balanced growth path (the long-run, steady-state growth path) capital, output, and consumption growth rates per worker in the Solow-Swan model are determined by the exogenous growth rate of technical progress (growth of 'knowledge'), g, and therefore are independent of the saving rate (and anything else). This result is both important and disappointing, in that we have a theory of growth that yields the conclusion that economic growth is driven by exogenous growth of technology, the sources of which are unexplained. Along with alleged inconsistencies of the Solow-Swan model with important stylized facts, this deficiency is a principal motivation of 'new growth theory', which aims to make $A(t)$ endogenous.

Yet the Solow-Swan model does have interesting implications about the convergence of economies to steady-states. In this section we investigate how $g_k \equiv \frac{\dot{k}}{k}$, $g_q \equiv \frac{\dot{q}}{q}$ and $g_c \equiv \frac{\dot{c}}{c}$ respond to movements in in the capital stock, k, along its transition to unique steady to steady-state equilibrium, $k^*(s)$.

5.1 *Capital Transition to Steady-State*

From the fundamental dynamic equation for capital accumulation we know

$$g_k = \frac{sf(k)}{k} - (\delta + g + n)\,. \tag{41}$$

Therefore

$$\begin{aligned} \frac{\partial g_k}{\partial k} &= \frac{sf'(k)}{k} - \frac{sf(k)}{k^2} \\ &= -s \cdot \left[\frac{(f(k) - f'(k) \cdot k)}{k^2} \right] < 0. \end{aligned} \tag{42}$$

The derivative is negative because $f'(k) \cdot k$ (income going to capital) is less than $f(k)$ (total income). It follows that k converges to k^* at a rate that declines with with k. [*Show graph*]

Since $\frac{\dot{q}}{q} = \alpha_k \frac{\dot{k}}{k}$, an implication is that "capital poor" economies should grow faster than "capital rich" economies *when* savings rates (and other given parameters of the model) are common.

Although the Solow-Swan model is lodged in a closed economy environment, it would seem to follow also that investment incentives should favor capital poor economies and, consequently, that capital (human and physical) should flow from rich to poor economies. Alas, by and large it does not when 'economies' are taken to be nations. In fact investment flows (particularly in the form of flows of high skill workers) in many cases go in the opposite direction – from economies with low capital endowments to economies with high endowments. I return to these issues later in the lectures on convergence and growth and development empirics. For now let us examine the transitional dynamics of output analytically.

5.2 *Output Transition to Steady-State*

We know already that $g_q = \alpha_k \cdot g_k$ and that $g_k = \frac{sf(k)}{k} - (\delta + g + n)$. Hence

$$g_q = \alpha_k \cdot \left[\frac{sf(k)}{k} - (\delta + g + n) \right] \tag{43}$$

which after substitution for α_k gives

$$g_q = \frac{f'(k)\cdot k}{f(k)}\cdot\left[\frac{sf(k)}{k}-(\delta+g+n)\right] \tag{44}$$
$$= sf'(k)-(\delta+g+n)\cdot\frac{f'(k)\cdot k}{f(k)}.$$

Now find how the output per effective worker growth rate, g_q, responds to movements in k :

$$\frac{\partial g_q}{\partial k} = \left\{\begin{array}{c} sf''(k)-(\delta+g+n)\cdot\frac{f''(k)\cdot k}{f(k)}-(\delta+g+n)\cdot\frac{f'(k)}{f(k)} \\ +(\delta+g+n)\cdot\frac{[f'(k)]^2\cdot k}{[f(k)]^2}\end{array}\right\} \tag{45}$$

$$= \left\{\begin{array}{c} f''(k)\cdot\left[s-(\delta+g+n)\cdot\frac{k}{f(k}\right] \\ -(\delta+g+n)\cdot\frac{f'(k)}{f(k)}\cdot\left[1-\frac{f'(k)\cdot k}{f(k)}\right]\end{array}\right\}$$

$$= \left\{\begin{array}{c} f''(k)\cdot\left[s-(\delta+g+n)\cdot\frac{k}{f(k}\right] \\ -(\delta+g+n)\cdot\frac{f'(k)}{f(k)}\cdot[1-\alpha_k]\end{array}\right\}$$

Multiplying the first term on the right-side by $\frac{k}{f(k)}\cdot\frac{f(k)}{k}(=1)$ we obtain

$$\frac{\partial g_q}{\partial k} = \left\{\begin{array}{c} f''(k)\cdot\frac{k}{f(k)}\cdot\left[s\cdot\frac{f(k)}{k}-(\delta+g+n)\right] \\ -(\delta+g+n)\cdot\frac{f'(k)}{f(k)}\cdot[1-\alpha_k]\end{array}\right\} \tag{46}$$

Since $g_k = s\frac{f(k)}{k}-(\delta+g+n)$,the equation above can be written

$$\frac{\partial g_q}{\partial k} = \left\{\begin{array}{c} f''(k)\cdot\frac{k}{f(k)}\cdot g_k \\ -(\delta+g+n)\cdot\frac{f'(k)}{f(k)}\cdot[1-\alpha_k]\end{array}\right\}. \tag{47}$$

What can we now say about the response of g_q to movements in k, which is a principal object of the exercise? The second right-side term in the equation above for $\frac{\partial g_q}{\partial k}$ is always < 0 because $f'(k) > 0$ and $0 < \alpha_k < 1$. So the sign of $\frac{\partial g_q}{\partial k}$is determined by the first right-side term.

In the first right-side term $f''(k) < 0$, by the assumption of diminishing returns (concavity) of the neoclassical production function. So the sign of the first term is determined by the sign of g_k. If capital is below the steady-state value implied by the prevailing savings rate, that is, if $\frac{sf(k)}{k} > (\delta + g + n)$ and therefore $k < k^*$, then $g_k > 0$. In this case the growth rate of output, g_q, unambiguously slows as the capital stock, k, rises and, therefore, g_q slows also as q rises.

If g_k is negative because $k > k^*$, then the sign of $\frac{\partial g_q}{\partial k}$ is indeterminate. If the economy is in the vicinity of $k = k^*$, however, we know g_k will be small which implies that the second right-side term in the $\frac{\partial g_q}{\partial k}$ function dominates, and so g_q falls as k and q rise.

5.3 *Consumption Transition to Steady-State*

Recall $c = (1 - s)q$. Hence

$$\begin{aligned} g_c &\equiv \frac{\dot{c}}{c} = \frac{(1-s)\dot{q}}{(1-s)q} \\ &= g_q. \end{aligned} \tag{48}$$

g_c and g_q therefore have identical transitional dynamics.

Finally remember that $\widetilde{x} = A(t) \cdot x,\ x \Rightarrow \{k, q, c\}\,,\ \frac{\dot{A}}{A} = g$, and therefore $g_{\widetilde{x}} = g_x + g$. Because technology is exogenous in the neoclassical model, the growth of $A(t)$ is not affected by capital formation (or anything else in the model environment). Hence the transitional dynamics of per worker growth rates $g_{\widetilde{x}}$, with respect to k are identical to the transitional dynamics of per effective worker growth rates g_x with respect to k.

6 The Speed of Convergence when k is near k^*

6.1 *Capital*

When k is in the vicinity of k^* we can obtain a useful approximation of the speed of convergence by taking a first-order Taylor expansion of $\dot{k} = sf(k) -$

$(\delta + g + n)\,k$ at $k \simeq k^*$.[5] The first-order approximation is

$$\dot{k} \simeq sf(k^*) - (\delta + g + n)\,k^* + [sf'(k^*) - (\delta + g + n)] \cdot (k - k^*) \tag{49}$$

which since $[sf(k^*) - (\delta + g + n)\,k^*] = 0$, simplifies to

$$\dot{k} \simeq [sf'(k^*) - (\delta + g + n)] \cdot (k - k^*)\,. \tag{50}$$

Hence for $g_k \equiv \frac{\dot{k}}{k}$ we obtain

$$g_k \simeq [sf'(k^*) - (\delta + g + n)] \cdot \frac{(k - k^*)}{k}. \tag{51}$$

Since the given rate of saving is $s_{(k^*)} = \frac{(\delta+g+n)k^*}{f(k^*)}$, the first-order approximation for g_k is

$$\begin{aligned} g_k &\simeq \left[\frac{(\delta + g + n)\,k^*}{f\,(k^*)} f'(k^*) - (\delta + g + n)\right] \cdot \frac{(k - k^*)}{k} \\ &\simeq [(\delta + g + n) \cdot (\alpha_{k^*} - 1)] \cdot \frac{(k - k^*)}{k} \\ &\simeq [(\delta + g + n) \cdot (1 - \alpha_{k^*})] \cdot \frac{(k^* - k)}{k} \end{aligned} \tag{52}$$

Because $\frac{(k^*-k)}{k}$ is very nearly equal to $(\ln k^* - \ln k)$ when k^* is not too far from k, the approximation to the growth rate of capital may be written:

$$g_k \simeq [(\delta + g + n) \cdot (1 - \alpha_{k^*})] \cdot (\ln k^* - \ln k)\,, \tag{53}$$

[5] Recall Taylor's nth-order polynomial expansion formula for approximating a function, here the fundamental dynamic equation for capital, $\dot{k}$, with finite derivatives of all orders when evaluated at some value, here $k = k^*$:

$$\dot{k} = \left\{ \begin{array}{c} sf(k^*) - (\delta + g + n)\,k^* + \frac{1}{1!}\frac{\partial \dot{k}}{\partial k^*} \cdot (k - k^*) \\ + \frac{1}{2!}\frac{\partial^2 \dot{k}}{(\partial k^*)^2} \cdot (k - k^*)^2 + ... + \frac{1}{n!}\frac{\partial^n \dot{k}}{(\partial k^*)^n} \cdot (k - k^*)^n \\ + R_{n+1} \end{array} \right\}$$

The remainder (due to Lagrange) is $R_{n+1}(x) = \frac{1}{(n+1)!}\frac{\partial^n \dot{k}}{(\partial x)^n} \cdot (k - k^*)^{n+1}$, where x is some number between k and k^*.

The growth rate of capital per effective worker, $g_k \equiv \frac{d \ln k}{dt}$, is a differential equation in $\ln k(t)$ which has solution

$$\begin{aligned} \ln k(t) &= \ln k^* + e^{-\beta t} \cdot [\ln k(0) - \ln k^*] \\ &= e^{-\beta t} \cdot \ln k(0) + \left(1 - e^{-\beta t}\right) \cdot \ln k^* \end{aligned} \tag{54}$$

where $k(0) > 0$ is an initial condition and β is a composite parameter that denotes $(\delta + g + n) \cdot (1 - \alpha_{k^*})$.[6]

Express the solution as $[\ln k(t) - \ln k^*] = e^{-\beta t} \cdot [\ln k(0) - \ln k^*]$, and let $gap_k(t)$ denote the left-side, that is, the deviation of log capital per effective worker, $\ln k(t)$, from its its steady-state, $\ln k^*$. Note that

$$\frac{d\left[gap_k(t)\right]}{dt} = -\beta \cdot gap_k(t) \tag{55}$$

$$\frac{1}{gap_k(t)} \frac{d\left[gap_k(t)\right]}{dt} = -\beta \tag{56}$$

Hence $\ln k(t)$ converges to its steady-state value $\ln k^*$ at a rate per unit time equal to β (now commonly know as "beta convergence"). The convergence rate, β, depends importantly on the magnitude of of α_k; the larger α_k the slower the implied rate of convergence to steady-state. This will become important later when we consider the consistency of the neoclassical model with data.

Looking at the solution in the form of the second line of (54),

$$\ln k(t\cdot) = e^{-\beta t} \cdot \ln k(0) + \left(1 - e^{-\beta t}\right) \cdot \ln k^*,$$

note that the 'half life' (the time date t at which $\ln k(t)$ is halfway from the initial condition $\ln k(0)$ to its steady-state destination value $\ln k^*$) satisfies the relations

$$e^{-\beta t} = \left(1 - e^{-\beta t}\right) = \frac{1}{2},$$

$$-\beta t = \ln \frac{1}{2}$$

[6] I work with log capital (and below with log output) because log quantities relate more naturally to neoclassical growth theory empircs, which we take up later. The results all pass through, however, to k and q. For example, the solution to the differential equation for $\frac{dk}{dt}$,

$\dot{k} \simeq [(\delta + g + n) \cdot (1 - \alpha_{k^*})] \cdot (k - k^*)$,

is

$k(t) = k^* + e^{-\beta t} \cdot [k(0) - k^*]$.

$$\beta t = \ln 2 \Longrightarrow t = \ln 2/\beta = 0.69/\beta$$

Therefore, if we had a reliable estimate of β we could calculate in the same fashion the time it would take the capital stock to converge to any fraction of its steady-state value, $\ln k^*$, from some starting date value, $\ln k(0)$.

6.2 ***Output***

The first-order approximation for the speed of output convergence shows that q approaches q^* at that same rate as capital approaches its steady-state value. This can be established using the same basic approach used above to obtain the convergence rate of capital. We know that $q = f(k)$, and that $\dot{k} = sq - (\delta + g + n)k$, and so

$$\begin{aligned} \dot{q} &= f'(k) \cdot \dot{k} \\ &= f'(k) \cdot [sq - (\delta + g + n)k]. \end{aligned} \tag{57}$$

Taking a first-order approximation around $q \simeq q^*$ we obtain

$$\dot{q} \simeq \left\{ \begin{array}{c} f'(k^*) \cdot [sq^* - (\delta + g + n)k^*] \\ + \frac{\partial\{f'(k^*) \cdot [(sq^* - (\delta+g+n)k^*)]\}}{\partial q^*} \cdot (q - q^*) \end{array} \right\} \tag{58}$$

$$\begin{aligned} \dot{q} &\simeq 0 + \left[s \cdot f'(k^*) - (\delta + g + n) \cdot f'(k^*) \cdot \left(\frac{\partial q^*}{\partial k^*} \right)^{-1} \right] \cdot (q - q^*) \\ &\simeq \left[s \cdot f'(k^*) - (\delta + g + n) \cdot f'(k^*) \cdot (f'(k^*))^{-1} \right] \cdot (q - q^*) \\ &\simeq [s \cdot f'(k^*) - (\delta + g + n)] \cdot (q - q^*). \end{aligned} \tag{59}$$

Substituting the given rate of saving $s_{(k^*,q^*)} = \frac{(\delta+g+n)k^*}{q^*}$ into the above equation yields

$$\begin{aligned} \dot{q} &\simeq \left[\frac{(\delta + g + n)k^*}{q^*} \cdot f'(k^*) - (\delta + g + n) \right] \cdot (q - q^*) \\ &\simeq [(\delta + g + n)\alpha_{k^*} - (\delta + g + n)] \cdot (q - q^*) \\ &\simeq [(\delta + g + n) \cdot (\alpha_{k^*} - 1)] \cdot (q - q^*) \end{aligned} \tag{60}$$

$$\dot{q} \simeq \beta \cdot (q^* - q) \tag{61}$$

where, as before, $\beta = [(\delta + g + n) \cdot (1 - \alpha_{k^*})]$.

The growth rate of output per effective worker, $g_q \equiv \frac{\dot{q}}{q} \equiv \frac{d \ln q}{dt}$, is therefore given by

$$g_q \simeq \beta \cdot \frac{(q^* - q)}{q} \tag{62}$$

$$g_q \simeq \beta \cdot (\ln q^* - \ln q) \tag{63}$$

where I make use of the fact that $\frac{(q^* - q)}{q} \simeq (\ln q^* - \ln q)$ when $\ln q$ is not too far from $\ln q^*$.

$g_q \equiv \frac{d \ln q}{dt}$ is a linear differential equation for $\ln q$ which for given initial condition $\ln q(0)$ has a solution in exactly the same form as the corresponding solution for $\ln k$:

$$\begin{aligned} \ln q(t) &= \ln q^* + e^{-\beta t} \cdot [\ln q(0) - \ln q^*] \\ &= e^{-\beta t} \cdot \ln q(0) + \left(1 - e^{-\beta t}\right) \cdot \ln q^* . \end{aligned} \tag{64}$$

When q is not too far from q^*, output therefore exhibits the same convergence properties as capital, no matter what the functional form of the underlying neo-classical production function and, consequently, earlier remarks about capital's rate of convergence apply here as well. Also remember that consumption obeys the same dynamics as output, and so its (identical) convergence properties can be deduced from:

$$\ln c(t) = \ln c^* + e^{-\beta t} \cdot [\ln c(0) - \ln c^*] . \tag{65}$$