Douglas Hibbs
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# Neoclassical Growth Theory with Endogenous Saving

(Ramsey-Cass-Koopmans)

The state of the world is the same as in the Bare Bones model except time is now continuous and production is governed by unspecified neoclassical technology. As before, the economy is populated by a large number of identical firms and households. Firms aim to maximize profits and they rent capital, hire labor and sell output at competitive prices in competitive markets. Households aim to maximize a Utility of intertemporal consumption function. They own the firms (the capital stock) and so profits accrue to them.

The model specifies saving behavior in terms of "deep" parameters of utility and preference. We will be able to show that the possibility of inefficient 'over-saving' (the "dynamically inefficient" region of the output-capital space) is ruled out when the macroeconomic rate of saving (consumption) is ground out by the optimal decisions of individual households.

## 1 Households

### 1.1 *The Utility Objective*

Households are dynastic with generations linked through time by altruistic bequests. They behave identically and seek to maximize the time-separable intertemporal utility function

$$
\begin{aligned}
U_{(t=0)} &= \int_{t=0}^{\infty} u\left[\widetilde{c}\,(t)\right] \cdot e^{nt} \cdot e^{-\rho t} dt \qquad (1) \\
&= \int_{t=0}^{\infty} u\left[\widetilde{c}\,(t)\right] \cdot e^{-(\rho-n)t} dt
\end{aligned}
$$

where household size, $N$, is given exogenously by $N(t) = e^{nt}$, $N(0) = 1$; $\widetilde{c} \equiv \frac{C}{N}$, denotes household consumption per (adult, working) member; the discount rate exceeds the growth rate of household size (necessary to achieve a convergent integral) $0 < \rho > n$. Instantaneous utility ("felicity") has properties

$$u'[\widetilde{c}] > 0, \quad u''[\widetilde{c}] < 0,$$

$$\underset{as \quad \widetilde{c}\to 0}{u'[\widetilde{c}] \to \infty}, \quad \underset{as \quad \widetilde{c}\to\infty}{u'[\widetilde{c}] \to 0}$$

$U_{(t=0)}$ is the present-value (discounted integral) of a (notional) stream of instantaneous period utilities ('felicities') of consumption per household member. $u[\widetilde{c}(t)] \cdot e^{nt} = u[\widetilde{c}(t)] \cdot N(t)$ gives the household's gross utility of consumption (that is the household's aggregate 'utils' of consumption) per period. Because households are identical, the setup gives aggregate or 'social' utility which maps naturally to aggregate consumption, saving, capital formation and growth.

## 1.2 *The Budget Constraint*

Each household member inelastically and successfully supplies one unit of labor to market per unit of time, which commands a market wage income of $w(t)$. Unlike the setup of the bare-bones model, we therefore abstract from optimal labor supply decisions. Households can borrow and lend at an interest rate given by the market, $r(t)$. Household income per adult member at each period is therefore

$$w(t) + r(t) \cdot a(t) - n \cdot a(t)$$

where $a$ is the stock of net assets or financial wealth per member; it may be negative or positive in a given period. $a(0)$ is given (bequests from earlier household generations). Everything is real and expressed in units of consumables. The household's budget constraint is[1]:

$$\begin{aligned} \dot{a}(t) &= w(t) + r(t) \cdot a(t) - n \cdot a(t) - \widetilde{c}(t) \\ &= w(t) + [r(t) - n] \cdot a(t) - \widetilde{c}(t). \end{aligned} \tag{2}$$

[1]Denote total household assets by $A$ (not to be confused with the state of technology), and per adult assets, $\frac{A}{N}$, by $a$. $A$ evolves

$$\dot{A} = rA + wN - C$$

where $N$ is total units of household labor and $C$ is total consumption.

Dividing by $N$ gives

$$\dot{A}/N = r \cdot a + w - \tilde{c}.$$

If households could borrow without limit at the market rate of interest, incentive would exist to pursue Ponzi scheme-chain letter strategies financing in perpetuity arbitrarily high consumption levels unsecured by present and future income flows. The budget constraint would therefore not bind. We assume that the credit market rules out such strategies by imposing a "no Ponzi scheme" restriction requiring that in the limit the present value of assets must be non-negative:

$$\lim_{t\to\infty}\left\{a(t)\cdot\exp\left[-\int_{\tau=0}^{t}[r(\tau)-n]\,d\tau\right]\right\}\succeq 0. \tag{3}$$

This restriction just means that debt per household member ($a(t)<0$) cannot in the long run grow faster than the per member rate of return to saving $[r(t)-n]$; in other words aggregate household debt, $N(t)\cdot a(t)$, cannot grow faster than $r(t)$. [See ahead for further evaluation of this expression.]

## 1.3 Solution Setup for the Household's Program

## 1.4 *The Hamiltonian*

We shall solve the household's constrained optimization problem by setting up a present-value Hamiltonian[2] (after William R. Hamilton, like so many greats, myself included, an Irishman, b. 1805 Dublin, d. 1865 Dublin):

$$\begin{aligned} H_{(t=0)} &= u[\widetilde{c}(t)]\cdot e^{-(\rho-n)t}+v(t)\cdot[\dot{a}(t)] \qquad (4)\\ &= u[\widetilde{c}(t)]\cdot e^{-(\rho-n)t}+v(t)\cdot[w(t)+[r(t)-n]\cdot a(t)-\widetilde{c}(t)] \end{aligned}$$

where $v(t)$ are continuous time Lagrangian multipliers giving the present-value shadow prices of wealth, that is, the value of an increment to assets at time-t,

Since $\dot{a}\equiv\dot{A}/N-A/N\cdot\frac{\dot{N}}{N}$, after substitution we get the budget constraint in the main text:

$$\dot{a}=r\cdot a+w-\tilde{c}-a\cdot n.$$

[2] There are other solution setups to well posed intertemporal optimization problems. You have already seen one (sequential discrete time Lagrange) in the DGME lecture. You will see another in the lectures on the life cycle-permanent income hypothesis (Bellman). They are of course intimately related.

measured in units of time-zero utils.[3] In the lingo of engineer's, $\widetilde{c}$ is the "control" (decision or choice) variable, $a$ (with $a(0)$ given) is the "state" variable, and $v$ is the "co-state" variable.

*Remarks:*

- $H$ can be thought of as the utility prospect of decisions about $\widetilde{c}\,(t)$ evaluated at decision period $t = 0$. For given shadow price $v(t)$, the Hamiltonian represents the total contribution to time-zero utility of a (notional) choice of $\widetilde{c}\,(t)$. The prospect is the sum of two components.

- The first right-side term, $u\,[\widetilde{c}\,(t)] \cdot e^{-(\rho-n)t}$, gives the time-zero utility value (the value today) of each constituent of a notional consumption stream. It obviously rises (falls) with a higher (lower) choice of $\widetilde{c}\,(t)$, but has an opportunity cost (benefit) given by the second right-side term.

- The second right-side term, $v(t) \cdot [\dot{a}\,(t)]$, gives the time-t value (calibrated in time-zero utils) of an increment to time-t wealth (which affects subsequent consumption possibilities) that is produced by a consumption choice $\widetilde{c}\,(t)$. In other words, the second-term is the value to the utility prospect of the change in assets produced by a consumption choice.

[*See H, $\widetilde{c}$ Graph ahead in the lecture*]

### 1.5 *The First Order Conditions*

Two FOCs and a terminal or "transversality" condition must be satisfied. The maximum principle (Pontryagin et al. 1962) requires

$$\frac{\partial H}{\partial \widetilde{c}\,(t)} = 0, \qquad \textit{all } t \tag{5}$$

that is

$$u'\,[\widetilde{c}\,(t)] \cdot e^{-(\rho-n)t} - v(t) = 0 \tag{6}$$

[3] Sometimes such problems are set up with current value shadow prices, in which case $v(t)$ is replaced by $q(t) = v(t) \cdot e^{(\rho-n)t}$. Current value shadow prices give the utility value of changes in wealth at time-t in time-t utils, as viewed from the household's time-zero perspective. Note also that the first right-side term in (4) could be written with shadow price $v(0)$, which by convention is normalized to unity.

implying that

$$v(t) = u'\left[\widetilde{c}\,(t)\right] \cdot e^{-(\rho-n)t}. \tag{7}$$

The first FOC indeed helps find a maximum because $\frac{\partial^2 H}{\partial[\widetilde{c}(t)]^2} = u''\left[\widetilde{c}\,(t)\right] < 0$ by assumption of the concavity of utility. The FOC implies that optimal behavior requires that the marginal utility of consumption at time-t, discounted back to the present time-zero, be equal to the present value (the value to time-zero marginal utility of consumption) of the associated change to assets at time-t. So the discounted increase in marginal utility owing to a choice of higher consumption (lower saving) must equal the marginal utility cost of the change in wealth $(-v(t))$ that is caused by this consumption choice.

The second FOC is:

$$\begin{aligned} \frac{\partial H}{\partial a\,(t)} + \dot{v}(t) &= 0 \\ \dot{v}(t) &= -\frac{\partial H}{\partial a\,(t)} \\ \dot{v}(t) &= -v\,(t) \cdot \left[r\,(t) - n\right] \end{aligned} \tag{8}$$

which is known as the Ramsey rule of optimal saving. It implies that the change in the value of an increment to time-t assets to time-zero marginal utility (the "capital gain" in marginal utils) must be offset exactly by the time-zero marginal utility value of the time-t return of an increment to assets (the "income" in time-zero marginal utils). Note the resemblance to investment behavior, where agents should value equally (tax considerations aside) an increment of capital gain and an increment of income.

Finally we have a transversality condition

$$\lim_{t\to\infty} \left[v(t) \cdot a\,(t)\right] = 0 \tag{9}$$

which means that the time-zero utility value of assets held at the end of the world is zero; either the shadow price, $v$, or asset stock, $a$, must go to zero asymptotically. If this were not the case there would be terminal "waste" in the family's optimal utility of consumption program.[4]

Integrating the result of the second FOC, $\dot{v}(t) = -v\,(t) \cdot \left[r\,(t) - n\right]$, over time gives

$$v\,(t) = v\,(0) \cdot \exp\left[-\int_{\tau=0}^{t} \left[r\,(\tau) - n\right] d\tau\right]. \tag{10}$$

[4] There is some controversy about the applicability of this condition in all situations. See the discussion and references in Barro and Sala-i-Martin; especially the 'Ramsey counter-example'.

Substituting $v(t)$ into the transversality condition, $\lim_{t\to\infty}[v(t)\cdot a(t)]=0$, allows us to express $\lim_{t\to\infty}[v(t)\cdot a(t)]=0$ as[5]

$$\lim_{t\to\infty}\left\{a(t)\cdot\exp\left[-\int_{\tau=0}^{t}[r(\tau)-n]\,d\tau\right]\right\}=0. \tag{11}$$

Note that if we evaluate the integral term $\int_{\tau=0}^{t} r(\tau)\,d\tau$ in terms of the average interest rate between 0 and $t$, namely

$$\overline{r}(t)=\frac{1}{t}\int_{\tau=0}^{t} r(\tau)\,d\tau,$$

then the present value exponential term is

$$\begin{aligned}\exp\left[-\int_{\tau=0}^{t}[r(\tau)-n]\,d\tau\right] &= e^{-[\overline{r}(t)-n]\cdot t} \\ &= \frac{1}{e^{[\overline{r}(t)-n]\cdot t}}.\end{aligned} \tag{12}$$

So the transversality condition implies that

$$\left\{\lim_{t\to\infty}\left[\frac{a(t)}{e^{[\overline{r}(t)-n]\cdot t}}\right]\right\}=0 \tag{13}$$

which means that wealth per household member, $a$, does not asymptotically grow as fast as returns to saving per member, $[\overline{r}(t)-n]$, or equivalently that aggregate household wealth, $N(t)\cdot a(t)\equiv e^{nt}\cdot a(t)$, does not grow as fast as aggregate returns to household saving, $e^{\overline{r}(t)\cdot t}$.

*Optimal Consumption-Saving:*

We obtain the Euler equation for optimal choice of consumption over time by differentiating the first FOC,

$$v(t)=u'[\widetilde{c}(t)]\cdot e^{-(\rho-n)t},$$

[5]Recall that $v(0)$ is set to unity by convention.

with respect to time, in order to find $\dot{v}$ :

$$\begin{aligned}
\frac{dv(t)}{dt} &= \frac{d\left[u'\left[\widetilde{c}\,(t)\right]\cdot e^{-(\rho-n)t}\right]}{dt} \qquad (14)\\
&= e^{-(\rho-n)t}\cdot\frac{d\left[u'\left[\widetilde{c}\,(t)\right]\right]}{dt}+u'\left[\widetilde{c}\,(t)\right]\cdot\frac{d\left[e^{-(\rho-n)t}\right]}{dt}\\
&= e^{-(\rho-n)t}\cdot\frac{d\left[u'\left[\widetilde{c}\,(t)\right]\right]}{d\left[\widetilde{c}\,(t)\right]}\cdot\frac{d\left[\widetilde{c}\,(t)\right]}{dt}+u'\left[\widetilde{c}\,(t)\right]\cdot\frac{d\left[e^{-(\rho-n)t}\right]}{dt}\\
&= e^{-(\rho-n)t}\cdot u''\left[\widetilde{c}\,(t)\right]\cdot\dot{\widetilde{c}}(t)-(\rho-n)\cdot u'\left[\widetilde{c}\,(t)\right]\cdot e^{-(\rho-n)t}\\
&= e^{-(\rho-n)t}\cdot\left[u''\left[\widetilde{c}\,(t)\right]\cdot\dot{\widetilde{c}}(t)-(\rho-n)\cdot u'\left[\widetilde{c}\,(t)\right]\right].
\end{aligned}$$

Next, substitute this result into the second FOC,

$$\dot{v}(t) = -v\,(t)\cdot\left[r\,(t)-n\right],$$

which yields

$$-v\,(t)\cdot\left[r\,(t)-n\right] = e^{-(\rho-n)t}\cdot\left[u''\left[\widetilde{c}\,(t)\right]\cdot\dot{\widetilde{c}}(t)-(\rho-n)\cdot u'\left[\widetilde{c}\,(t)\right]\right]. \qquad (15)$$

Now substitute in for $v(t) = u'\left[\widetilde{c}\,(t)\right]\cdot e^{-(\rho-n)t}$ from the first FOC:

$$\left\{\begin{array}{c} -\left(u'\left[\widetilde{c}\,(t)\right]\cdot e^{-(\rho-n)t}\right)\cdot\left[r\,(t)-n\right] = \\ e^{-(\rho-n)t}\cdot\left[u''\left[\widetilde{c}\,(t)\right]\cdot\dot{\widetilde{c}}(t)-(\rho-n)\cdot u'\left[\widetilde{c}\,(t)\right]\right]\end{array}\right\} \qquad (16)$$

Getting rid of the exponentials on each side:

$$\left\{\begin{array}{c} -u'\left[\widetilde{c}\,(t)\right]\cdot\left[r\,(t)-n\right] = \\ \left[u''\left[\widetilde{c}\,(t)\right]\cdot\dot{\widetilde{c}}(t)-(\rho-n)\cdot u'\left[\widetilde{c}\,(t)\right]\right]\end{array}\right\}. \qquad (17)$$

Considerable intuition is gained by solving this equation for the interest rate[6], $r(t)$:

$$\begin{aligned}
r\,(t) &= n+\frac{\left[u''\left[\widetilde{c}\,(t)\right]\cdot\dot{\widetilde{c}}(t)-(\rho-n)\cdot u'\left[\widetilde{c}\,(t)\right]\right]}{-u'\left[\widetilde{c}\,(t)\right]} \qquad (18)\\
&= \rho-\frac{u''\left[\widetilde{c}\,(t)\right]\cdot\dot{\widetilde{c}}(t)}{u'\left[\widetilde{c}\,(t)\right]}
\end{aligned}$$

[6] This result is implicit in F.P. Ramsey, *Economic Journal* (1928), a piece of genuine genius that did not (re)enter economics until a generation later – in O.Ekstein, *Quarterly Journal of Economics* (1957).

Multiplying the right-side by $\frac{\widetilde{c}(t)}{\widetilde{c}(t)}$ (=1) we arrive at a famous equation:

$$r(t) = \rho - \left[\left(\frac{\widetilde{c}(t)}{u'[\widetilde{c}(t)]}\right) \cdot u''[\widetilde{c}(t)]\right] \cdot \left(\frac{\dot{\widetilde{c}}(t)}{\widetilde{c}(t)}\right). \tag{19}$$

This is an *important* result. The right-side term in brackets is the elasticity of marginal utility of consumption (the proportional response of marginal utility to a proportional change in consumption). Optimal behavior therefore implies that households choose a consumption time plan such that consumption per member at each period in the infinite planning horizon, $\widetilde{c}(t)\}_0^\infty$, equates the given market rate of interest $r(t)$ (the rate at which assets accumulate from saving), to their rate of time preference, $\rho$, plus the proportional change in marginal utility of consumption, $\left[\left(\frac{\widetilde{c}(t)}{u'[\widetilde{c}(t)]}\right) \cdot u''[\widetilde{c}(t)]\right]$, owing to proportional changes in consumption per household member, $\frac{\dot{\widetilde{c}}(t)}{\widetilde{c}(t)}$.

Note that left-side of the above equation is the rate of return to saving, whereas the right-side can be interpreted as a rate of return to consumption. At the optimum, households are indifferent between (and therefore equate) the two.

### 1.5.1 *The Utility Function*

Consider the above result for $r(t)$. Note that if the rate of time preference equals the interest rate (the price or time value of money), $\rho = r(t) = \overline{r}$, then $\frac{\dot{\widetilde{c}}(t)}{\widetilde{c}(t)}$ must be zero; households will optimally choose a flat consumption time path.(As we shall see in subsequent lectures, the flat consumption time path may equal the annuity value of wealth; a result forming the core of Milton Friedman's famous Permanent Income Hypothesis).

For $\frac{\dot{\widetilde{c}}(t)}{\widetilde{c}(t)}$ to be constant for $r(t)$ held at some value $\neq \rho$, the elasticity of marginal utility,$\left(\frac{\widetilde{c}(t)}{u'[\widetilde{c}(t)]}\right) \cdot u''[\widetilde{c}(t)]$, must be constant. Hence the popularity of the Constant Relative Risk Aversion (CRRA) or Constant Intertemporal Elasticity of Substitution (CIES) utility function:

$$u(\widetilde{c}) = \frac{\widetilde{c}^{1-\frac{1}{\sigma}}}{\left(1-\frac{1}{\sigma}\right)}, \qquad \sigma > 0, \sigma \neq 1. \tag{20}$$

Under CIES marginal utility is[7]

[7] At $\sigma = 1$, $u(\widetilde{c})] = \ln \widetilde{c}$ by l'Hôpital's rule, although to generate the proof you need to write

$$u'(\widetilde{c}) = \widetilde{c}^{-\frac{1}{\sigma}} \tag{21}$$

and the elasticity of marginal utility is constant, and is equal to $-\frac{1}{\sigma}$ :

$$\frac{\widetilde{c}}{u'(\widetilde{c})} \cdot u''(\widetilde{c}) = \frac{\widetilde{c}}{\widetilde{c}^{-\frac{1}{\sigma}}} \cdot \left( -\frac{1}{\sigma} \widetilde{c}^{-\frac{1}{\sigma}-1} \right) = -\frac{1}{\sigma}. \tag{22}$$

Under CIES utility, optimal consumption condition

$$r(t) = \rho - \left[ \left( \frac{\widetilde{c}(t)}{u'[\widetilde{c}(t)]} \right) \cdot u''[\widetilde{c}(t)] \right] \cdot \left( \frac{\dot{\widetilde{c}}(t)}{\widetilde{c}(t)} \right),$$

therefore turns out to be

$$r(t) = \rho - \left( -\frac{1}{\sigma} \right) \cdot \left( \frac{\dot{\widetilde{c}}(t)}{\widetilde{c}(t)} \right) \tag{23}$$

$$= \rho + \frac{1}{\sigma} \cdot \left( \frac{\dot{\widetilde{c}}(t)}{\widetilde{c}(t)} \right). \tag{24}$$

Consequently, the household's optimal consumption growth rate is

$$\frac{\dot{\widetilde{c}}(t)}{\widetilde{c}(t)} = \sigma \cdot [r(t) - \rho]. \tag{25}$$

As we saw earlier in the DGME lecture, the optimal time path of consumption is therefore determined by the gap between the interest rate and the rate of time preference, weighted by the intertemporal elasticity of substitution (the inverse of the negative of the elasticity of marginal utility). For a given gap between $r(t)$ and $\rho$, the bigger is the propensity to substitute consumption intertemporally (the larger is $\sigma$ ), the bigger is the response of the optimal time path of consumption.[8] It stems from the fact that the bigger is $\sigma$ the bigger is the gain (the smaller is the decline) in marginal utility generated by a proportional rise in consumption and,

CIES utility in the first instance as $u(\widetilde{c}) = \frac{\widetilde{c}^{1-\frac{1}{\sigma}}-1}{\left(1-\frac{1}{\sigma}\right)}$, otherwise the " -1" plays no role, and usually is omitted from $u(\widetilde{c})$, as throughout this lecture.

[8] An elasticity of substitution is a pure number measuring the proportional rate at which (in the present case) $\widetilde{c}(t2)$ is substituted for $\widetilde{c}(t1)$ (or subsitution is made between consumption at any pair of adjacent periods) in response to a proportional change in utility "prices" (equal to utility marginal 'products' of consumtpion,that is, marginal utilities) as the time interval $t1 \rightarrow t2$ becomes arbitrarily small (the limit, so that time discounting may be neglected). The elasticity

consequently, the more willing are households to deviate from a flat consumption time path.

If $r$ is constant at $\overline{r}$ over some interval from say $t = 0$ to $t = t$, then integrating the optimal solution for $\frac{\dot{\widetilde{c}}(t)}{\widetilde{c}(t)}$ gives the consumption time path

$$c(t) = c(0) \cdot e^{\sigma(\overline{r}-\rho)\cdot t}. \tag{26}$$

# 2 Firms

Production is neoclassical with the usual CRS properties

$$Q = F[K, (AN)] \tag{27}$$

and where as before

$$A(t) = A(0)e^{gt}, \qquad A(0) = 1. \tag{28}$$

*Marginal Productivities:*

As in earlier lectures $q$ and $k$ denote intensive form variates: $q \equiv \frac{Q}{A(t)\cdot N}$, $k \equiv \frac{K}{A(t)\cdot N}$. We already know that the marginal productivity of aggregate capital and intensive capital are equal

$$F'(K) = f'(k),$$

and so that capital's share in basic and intensive form are naturally identical

$$\frac{F'(K)\cdot K}{Q} = \frac{f'(k)\cdot k}{q},$$

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of substitution can be expressed

$$-\frac{d\ln\left[\widetilde{c}(t2)/\widetilde{c}(t1)\right]}{d\ln\left[u'\widetilde{c}(t2)/u'\widetilde{c}(t1)\right]}$$

which given CIES utilitiy is

$$\begin{aligned} -\frac{d\ln\left[\widetilde{c}(t2)/\widetilde{c}(t1)\right]}{d\left\{\ln\left[\widetilde{c}(t2)^{-\frac{1}{\sigma}}/\widetilde{c}(t1)^{-\frac{1}{\sigma}}\right]\right\}} &= -\frac{d\ln\left[\widetilde{c}(t2)/\widetilde{c}(t1)\right]}{-\frac{1}{\sigma}d\left[\ln\left[\widetilde{c}(t2)/\widetilde{c}(t1)\right]\right]} \\ &= \frac{1}{(1/\sigma)} = \sigma. \end{aligned}$$

and that by CRS (degree 1 homogeneity of production) we can write

$$Q = A(t) \cdot N \cdot f(k)], \tag{29}$$

so that labor's marginal product is

$$\begin{aligned} \frac{\partial Q}{\partial N} &= A(t) \cdot f(k) + A(t) \cdot N \cdot \frac{\partial f(k)}{\partial k} \cdot \frac{\partial k}{\partial N} \\ &= A(t) \cdot f(k) - A(t)N \cdot f'(k) \cdot \left(-\frac{K}{A(t) \cdot N^2}\right) \\ &= A(t) \cdot [f(k) - k \cdot f'(k)] \\ &= e^{gt} \cdot [f(k) - k \cdot f'(k)] \,. \end{aligned} \tag{30}$$

The marginal product of labor, and hence the wage, therefore grow with the state of technology, that is with the (exogenous) engine of economic growth. Note the same is not true of the return to capital,

$$\frac{\partial Q}{\partial K} = f'(k) \,. \tag{31}$$

It could not be otherwise; wages have grown secularly under market capitalism whilst the returns to capital exhibit no such secular increases.

## 2.1 *Profit*

The firm's profit per unit time is

$$\pi = Q - [r(t) + \delta] \cdot K - w \cdot N \tag{32}$$

where the user cost of capital is the market interest rate plus the rate of capital depreciation per unit time.

Substituting in $Q = A(t) \cdot N \cdot f(k)]$ and $w = \frac{\partial Q}{\partial N} = A(t) \cdot [f(k) - k \cdot f'(k)]$, we obtain

$$\begin{aligned} \pi &= A(t) \cdot N \cdot f(k) - [r(t) + \delta] \cdot K - A(t) \cdot [f(k) - k \cdot f'(k)] \cdot N \\ & A(t) \cdot N \cdot \left\{ \begin{array}{c} f(k) - [r(t) + \delta] \cdot k \\ -[f(k) - k \cdot f'(k)] \end{array} \right\} \\ &= A(t) \cdot N \cdot k \cdot \{f'(k) - [r(t) + \delta]\} \\ &= 0. \end{aligned} \tag{33}$$

We get the standard result that profits are zero in competitive equilibrium because capital will be deployed up to the point at which its marginal product, $f'(k)$, equals its user cost, $[r(t)+\delta]$. [*Discuss implications for dynamics of capitalism; relentless search for activities yielding transitory monopoly profits*]

# 3 Equilibrium

We now join the results for households and firms to analyze the competitive equilibrium. Remember everything is real and priced in units of consumables and that households own the capital stock. So we can equate assets per adult (per worker) in the household sector to capital per worker in the firm sector:

$$a(t) \equiv \frac{Assets\,(t)}{N(t)} = \frac{K(t)}{N(t)} = \widetilde{k}(t) = k(t) \cdot A(t) = k(t) \cdot e^{gt}. \tag{34}$$

The equilibrium is established by solving a pair of ordinary differential equations (ODEs); one for intensive form capital, the other for intensive form consumption. We know that capital evolves as

$$\dot{k} = sf(k) - (\delta + g + n)\,k$$

.Since $sf(k) = f(k) - c$ (investment per effective worker equals output per effective worker not consumed), the ODE for capital written in terms of consumption is

$$\dot{k} = f(k) - c - (\delta + g + n) \cdot k. \tag{35}$$

[*Graph: Dynamics of* $k$ *in* $c, k$ *space*]

From the solution to the household's optimization problem, we earlier derived the famous result (conditioned on the CIES utility function)

$$\frac{\dot{\widetilde{c}}(t)}{\widetilde{c}(t)} = \sigma \cdot [r(t) - \rho]$$

Given our parameterization of the evolution of technical progress, $A(t) = e^{gt}$, we know that

$$\frac{\dot{c}}{c} = \frac{\dot{\widetilde{c}}(t)}{\widetilde{c}(t)} - g. \tag{36}$$

And from the optimal behavior of firms we know that

$$f'(k) = [r(t) + \delta]$$

and therefore

$$r(t) = f'(k) - \delta$$

Hence the ODE for consumption in intensive (per effective worker) form is:

$$\begin{aligned} \frac{\dot{c}}{c} &= \frac{\dot{\widetilde{c}}(t)}{\widetilde{c}(t)} - g \\ &= \sigma \cdot [f'(k) - \delta - \rho] - g \\ &= \sigma \cdot \left[f'(k) - \delta - \rho - \frac{g}{\sigma}\right] \end{aligned} \tag{37}$$

[*Graph: Dynamics of* $c$ *in* $c, k$ *space*]

To find an equilibrium we also will need the transversality condition, written earlier as

$$\lim_{t\to\infty} \left[\frac{a(t)}{e^{[\overline{r}(t)-n]\cdot t}}\right] = 0.$$

After making the substitutions

$$\begin{aligned} a(t) &= A(t) \cdot k(t) \\ &= e^{gt} \cdot k(t) \\ &= \frac{k(t)}{e^{-gt}} \end{aligned}$$

and

$$\overline{r}(t) = f'(k) - \delta$$

we can write the transversality condition as

$$\lim_{t\to\infty} \left[\frac{k(t)}{e^{[f'(k)-\delta-n-g]\cdot t}}\right] = 0. \tag{38}$$

The result says that asymptotically the market rate of interest (the marginal product of capital less the depreciation rate) exceeds the growth rate of $K$ (that is, exceeds $(n+g)$).[9] Hence as $t \to \infty$, the denominator goes to infinity whereas

[9] Recall the steady-state result for the growth rate of capital per worker $\left[g_{\widetilde{k}^*} \,|\, (k = k^*)\right]$ in the Solow-Swan lecture. The steady-state growth of rate of $K$ is just $g_{\widetilde{k}^*} + n$, that is, $(g+n)$.

the numerator is finite, converging to $k^*$. Put another way, at steady-state it must be true that

$$f'(k^*) > (\delta + n + g) . \tag{39}$$

This condition becomes important just below, when we evaluate the steady-state of the system, and it distinguishes endogenous saving behavior from the "golden rule" saving rate implied by the Solow-Swan model.

## 3.1 *Steady State*

The ODEs for consumption and capital imply the following phase diagram. [*Graph phase diagram $\dot{c}$ and $\dot{k}$ in the $c, k$ space*] One can see that there is a unique equilibrium ("**E**" in the graph).

*Remarks:*

- The equilibrium exists and is unique. The steady-state growth rates $g_{k^*}$, $g_{c^*}$ (and $g_{q^*}$) are zero and the steady-state values of $k^*$, $c^*$ (and therefore $q^*$) are constant, just as in the Solow-Swan exogenous saving neoclassical model. So the Solow-Swan result did not hinge on the saving rate being given exogenously. Remember that this means that at steady-state the corresponding per worker variables – $\widetilde{k}$, $\widetilde{c}$, and $\widetilde{q}$ – grow at rate $g$ (the exogenous growth rate of technological progress), and that the aggregate level variables $K$, $C$ and $Q$ grow at rate $(n + g)$.

- The steady-state condition $\dot{k} = f(k^*) - c - (\delta + g + n) k^* = 0$, means that steady-state consumption is

$$c^* = f(k^*) - (\delta + g + n) k^* \tag{40}$$

and since $s = \left(1 - \frac{c}{f(k)}\right)$ that the steady-state saving rate is

$$s^* = \frac{(\delta + g + n) k^*}{f(k^*)} \tag{41}$$

and that steady-state output is of course

$$q^* = f(k^*) . \tag{42}$$

These results mimic Solow-Swan and therefore deliver no value added. And, as in Solow-Swan, to say something more explicit we need to specify $f(\cdot)$, the functional form of neoclassical production. [See the subsequent lecture on relative incomes under Cobb-Douglas production.]

- However, value added is delivered by the fact that $\left[f'(k) - \delta - \rho - \frac{g}{\sigma}\right] = 0$ at at steady-state, that is at $\dot{c} = 0$. (See eq. 37) Hence we arrive at a condition that defines what is sometimes called the "modified golden rule":

$$f'(k^*) = \delta + \rho + \frac{g}{\sigma}. \tag{43}$$

It follows that the saving rate and the associated steady-state capital stock when consumption-saving choices are made by optimizing households with a positive rate of time preference will be less than the "golden-rule" saving rate and capital stock. Therefore we will never observe 'dynamically inefficient' over-saving in steady-state (as in the prototypical 'Stalinist central planning' profile in which the economy is directed so as to amass huge stocks of capital at the expense of sustainable current consumption). Recall from the Solow-Swan setup (and observe in the phase graph) that the golden-rule is defined by a saving rate and steady-state capital stock satisfying $f'(k^*gold) = (\delta + g + n)$. By contrast in the endogenous saving neoclassical model

$$f'(k^*) = \left(\delta + \rho + \frac{g}{\sigma}\right) > (\delta + g + n) \tag{44}$$

which holds by the transversality condition in eq.(38); in particular in the denominator of (38) we had $f'(k) - (\delta + g + n) > 0$, as $t \to \infty$ and, therefore, as $k \to k^*$. Hence, in the endogenous saving marginal product of equilibrium capital, $f'(k^*) = \delta + \rho + \frac{g}{\sigma}$, is bigger than $f'(k^*gold) = (\delta + g + n)$. By the concavity of neoclassical production, it must follow that the endogenous saving, steady-state capital stock is smaller than the exogenous saving, golden rule steady-state capital stock.

The upshots of $f'(k^*) > f'(k^*gold)$ in the present endogenous saving set-up are:

$$k^*(s^*) < k^*gold\,(s^*gold) \tag{45}$$
$$q^*(k^*) < q^*gold\,(k^*gold) \tag{46}$$
$$c^*(s^*, q^*) < c^*gold\,(s^*gold, q^*gold) \tag{47}$$
$$s^* < s^*gold. \tag{48}$$

The results above stem from the fact that optimizing households are "impatient" and discount future consumption benefits at effective rate $\rho + \frac{g}{\sigma}$. Consequently they are unwilling to save enough (to defer enough current consumption) to obtain a higher level of sustainable consumption, $c^*gold$, which could be reached if consumption was without time value.

Finally, note that it is difficult to say much about the saving rate during the transition to steady-state. Even if we commit, say, to a Cobb-Douglas production function and a CIES utility function, the transition behavior of saving (and, therefore, the convergence behavior of the model) will depend on the magnitude of the utility parameter $\sigma$ in relation to that of the production parameter $\alpha$. Barro and Sala-i-Martin supply a good discussion of this point (and, for that matter, all the others).

**Lecture References and Remarks**

A very appealing treatise on the economics and technics of dynamic intertemporal optimization in setups, like the one in these lectures, is Kenneth Arrow and Mordecai Kurz, *Public Investment, The Rate of Return and Optimal Fiscal Policy* (1970, out of print). Another wonderful piece, packed with economic intuition is Robert Dorfman, 'An Economic Interpretation of Optimal Control Theory' *American Economic Review* (1969, Errata 1970). But my view is undoubtedly colored by nostalgia; Dorfman and Arrow and Kurz are what I studied when I was around your age. As is true of essentially all of the topics they cover, the treatment of the endogenous saving neoclassical model in Barro and Sala-i-Martin is a *tour de force.*