

Optimal Investment

1 Why Care About Investment?

- Investment drives capital formation, and the stock of capital is a key determinant of output and consequently feasible consumption levels.

$$K_{t+1} - K_t = I_t - \delta K_t \quad (1)$$

$$K_{t+1} = (1 - \delta) K_t + I_t \quad (2)$$

$$K_{t+1} = (1 - \delta)^{t+1} K_0 + \sum_{j=0}^t (1 - \delta)^j I_{t-j}. \quad (3)$$

- Investment is 15% or more of GDP and it is the most volatile component of GDP. Investment therefore may be an important proximate source of business cycles.¹
- Government policy is typically targeted heavily on investment; most tax codes favor it.
- Investment dynamics yield information about investor rationality, Keynes' "animal spirits," capital formation institutions, and more.

¹Recall the data in first Consumption lecture on the volatility of consumer durable purchases (consumer investment goods).

2 Static Neoclassical Investment Theory

Neoclassical investment theory dates mainly from Dale Jorgensen's papers in the 1960s (AER, 1963, Hall and Jorgenson, AER, 1967). However, what Jorgenson delivered was more a theory of the optimal capital stock than a theory of optimal investment per se. Here is the essential setup:

Output is produced by a CRS production technology

$$Q = F(K, A \cdot N) \quad (4)$$

Firms invest to achieve a capital stock that maximizes net real profit π each period

$$\pi_t = F(K_t, A \cdot N_t) - r_{Kt}K_t - w\bar{N} \quad (5)$$

where everything is priced in units of output, $r_K = p_K \cdot \left(r + \delta - \frac{\Delta p_K}{p_K} \right)$ is the real user cost of capital given the real price of capital units p_K , a fixed market interest rate r , and depreciation rate δ ,² and w is the real wage. We shall assume that labor input cost is given exogenously at $w\bar{N}$ (along with other input costs).

The necessary FOC for the desired stock of capital yields the implicit demand function

$$\begin{aligned} \frac{\partial \pi}{\partial K} &= F'K - r_K = 0 \\ \implies F'K &= r_K \end{aligned} \quad (6)$$

which is the usual neoclassical condition that the marginal productivity of the factor input deployed in production must equal its price.

Differentiating (6) wrt r_K to find the response of desired capital, K^D , to real

²The real user cost equals the real interest rate times real price of a unit of K plus the depreciation rate times real price of K minus the real shadow capital gain from purchasing a unit of K (positive if the price of K falls ex-post, negative if the price of K rises ex-post):

$$\begin{aligned} r_K &= r \cdot p_K + \delta \cdot p_K - \Delta p_K \\ &= p_K \cdot \left(r + \delta - \frac{\Delta p_K}{p_K} \right). \end{aligned}$$

user cost, r_K , we obtain

$$\begin{aligned}
\frac{d(F'K)}{dr_K} &= \frac{\partial [F'K - r_K]}{\partial r_K} \cdot \frac{dr_K}{dr_K} = 0 \\
&= \frac{\partial (F'K)}{\partial K} \cdot \frac{dK}{dr_K} - \frac{dr_K}{dr_K} = 0 \\
&= F''K \cdot \frac{dK}{dr_K} - 1 = 0 \\
\Rightarrow \frac{dK}{dr_K} &= \frac{1}{F''(K)} < 0
\end{aligned} \tag{7}$$

because of the concavity of the production function. So the desired stock of capital, K^D , is naturally downward sloping in the user cost of capital (in the real interest rate). [*Graph* ($F'K = r_K$), K^D demand curve] By Hotelling's lemma

$$\frac{\partial \pi}{\partial r_K} = -K^D | (K > 0) \tag{8}$$

so the producer is better off at the southeast end of the demand curve.

2.1 Implications for Investment Behavior

Since $K_{t+1} - K_t = I_t - \delta K_t$,

if $K^D > K$,

then $(I - \delta K) > 0$ and investment increases, and conversely.

At $K^D = K$, then $I = \delta K$, and investment is undertaken only to offset depreciation. (9)

[*Graph quadratic relation* π, K maxed at ($F'K = r_K$) giving instant adjustment of K to K^D]

2.2 Illustration with Cobb-Douglas

$$Q = AK^\alpha N^{1-\alpha} \tag{10}$$

The static FOC is

$$\begin{aligned}
F'K &= \alpha AK^{\alpha-1}N^{1-\alpha} = r_K \\
\alpha QK^{-1} &= r_K.
\end{aligned}
\tag{11}$$

Hence the conditional, desired stock of capital (conditional on the level of output) is

$$K^D = \frac{\alpha Q}{r_K}.$$

Jorgenson proposed in ad-hoc fashion a mechanism for the adjustment of actual to desired capital stock (because of unspecified “costs”) along the lines of

$$\begin{aligned}
(K_{t+1} - K_t) &= (1 - \gamma)(K_t^D - K_t), \quad \gamma \in (0, 1) \\
\implies K_{t+1} &= (1 - \gamma) \sum_{j=0}^{\infty} \gamma^j K_{t-j}^D.
\end{aligned}
\tag{13}$$

Given (12) the observed capital stock would be

$$K_{t+1} = (1 - \gamma) \sum_{j=0}^{\infty} \gamma^j \frac{\alpha Q_{t-j}}{r_{K_{t-j}}}.$$

The capital accumulation equation (1) implies that

$$I_t = K_{t+1}^D - (1 - \delta) K_t$$

from which it follows given (13) that investment is driven by

$$I_t = (1 - \gamma) \sum_{j=0}^{\infty} \gamma^j \frac{\alpha Q_{t-j}}{r_{K_{t-j}}} - (1 - \delta) K_t.$$

An equation of the sort that became known as a “flexible accelerator neoclassical model.”

Remarks:

- The static neoclassical setup implies that investment needed to get K to K^D occurs all at once – in one discrete jump. In general this would seem highly

unrealistic.

- Jorgenson has actual output on the right-side of the investment equation. This is *wrong*. What should appear is desired output, Q^D .
- 1960s neoclassical theory is pre-rational expectations and, therefore, expectations of future profitability of increments to the capital stock today play no role in explaining today's investment behavior.

3 Optimal Investment with Convex Adjustment Costs

The State of the World:

The firm is a price taker in competitive markets. Labor is flexible (can be adjusted without cost). The only barrier to full and fast deployment of the profit maximizing stock of capital are the convex costs of adjusting the capital stock. Firms can always obtain the capital desired (the market for investment capital is perfect). There are no taxes. The production technology has CRS. As we shall see, introducing adjustment costs yields rich dynamics to investment behavior.

3.1 The Producer's Objective

The representative firm aims to maximize its financial value, defined as the expected present discounted value of its net real revenue flow – revenue from sales less capital investment costs and labor (and other) costs

$$\text{Max}_{\{I_{t+j}, K_{t+j+1}\}_{j=0}^{\infty}} E_t \sum_{j=0}^{\infty} \beta^j \pi_{t+j} \quad (17)$$

$$\pi_t = F[K_t, (A \cdot N_t)] - I_t - C(I_t, K_t) - w_t N_t \quad (18)$$

subject to

$$K_{t+1} - K_t = I_t - \delta K_t \quad (19)$$

implying that investment satisfies

$$I_t = K_{t+1} - (1 - \delta) K_t \quad (20)$$

where I_t is the real cost of capital goods, $C(I_t, K_t)$ represents real adjustment costs in the form of diminished revenue from output sales, and $\beta = \frac{1}{(1+r)}$, $r > 1$, is the discount factor.

In order to get transparently to the famous “Tobin’s q ,”³ we set up the producer’s problem as a current value Lagrangian function

$$L = \underset{\{I_{t+j}, K_{t+j+1}\}_{j=0}^{\infty}}{Max} E_t \sum_{j=0}^{\infty} \beta^j \left\{ \begin{array}{l} F[K_{t+j}, (A \cdot N_{t+j})] - I_t - C(I_{t+j}, K_{t+j}) \\ -w_{t+j} N_{t+j} + q_{t+j} \cdot [I_{t+j} - K_{t+j+1} + (1 - \delta) K_{t+j}] \end{array} \right\} \quad (21)$$

where E_t denotes conditional expectations informed by all outcomes at period t and earlier, and q_t (q_{t+j}) gives the value (shadow price) of a marginal change in K at time $t + 1$ ($t + 1 + j$) in time t ($t + j$) net revenue.⁴

3.2 The FOCs

Because labor can be adjusted without cost, we need two FOCs $\implies I_t$ and K_{t+1} .

³After James Tobin, the great Yale University economist, who passed in 2002. Like many of the postwar Keynesian giants, he was raised during the Great Depression, which motivated his interest in economics in order to help put things right. Investment analysis via Tobin’s q (JMCB, 1969) was an important motivation for his 1981 Nobel Memorial prize in economics. He also won the Clark Medal in 1955 (until recently harder to win than the Nobel many Americans will argue). Father of the “Tobit” estimator in econometrics (see ahead on the name’s origin), among many other outstanding scientific accomplishments. Advocate of the “Tobin tax” to discourage speculative financial flows (“hot money”). His wartime service as a young officer on a US Navy destroyer was the inspiration for the sympathetic character Lt. Tobit in Herman Wouk’s novel *The Caine Mutiny*. What a life!

⁴Not to be confused obviously with my repeated use of “ q ” in the growth theory lectures to denote real output per effective worker.

The Investment FOC

$$\frac{\partial L}{\partial I_t} = -1 - C'(I_t) + q_t = 0. \quad (22)$$

Hence we obtain the shadow price of a new unit of capital as 1.0 (the normalized price of a unit of capital) plus the marginal cost of its installation:

$$q_t = 1 + C'(I_t) \quad (23)$$

$$\text{Marginal Benefit } (q_t) = \text{Marginal Cost } (1 + C'(I_t))$$

To be more explicit we need to specify the cost function, $C(I, K)$. A common specification is the strictly convex quadratic function

$$C(I_t, K_t) = \frac{b}{2} \left(\frac{I_t}{K_t} - \bar{C} \right)^2 \cdot K_t \quad (24)$$

where \bar{C} is the ‘costless’ level of installation (which could of course be zero) and b is just a scale parameter. [*Graph Cost function*]

In this fairly flexible case the shadow price is

$$\begin{aligned} q_t &= 1 + C'(I_t) \\ &= 1 + b \left(\frac{I_t}{K_t} - \bar{C} \right). \end{aligned} \quad (25)$$

As initially proposed by Tobin (JMCB, 1969), q is therefore a sufficient statistic for investment. The FOC for I_t in (25) implies the investment equation (investment relative to the existing stock of capital)

$$\frac{I_t}{K_t} = \frac{1}{b} \cdot (q_t - 1) + \bar{C}. \quad (26)$$

This is a famous equation, so be sure you understand it.⁵ It means that if the firm is maximizing it should invest up to the point at which the marginal cost of a new unit of capital equals its value (shadow price) to the PDV of its net real revenues. When the value of a new unit of capital exceeds its price (1.0), investment should rise above the costless level \bar{C} , and of course conversely.

3.2.1 Capital Stock Dynamics

Since

$$\Delta K_{t+1} = I_t - \delta K_t$$

$$\frac{\Delta K_{t+1}}{K_t} = \frac{1}{b} \cdot (q_t - 1) + (\bar{C} - \delta) \quad (27)$$

[Graph of ΔK_{t+1} at $\bar{C} = \delta$, assuming it is costless to replace depreciated capital]

3.2.2 The Capital FOC

The FOC is found for K_{t+1} because that is what is driven by I_t . (K_t is sunk.)⁶

$$\frac{\partial L}{\partial K_{t+1}} = E_t \left\{ \begin{array}{l} \beta \cdot [F'(K_{t+1}) - C'(K_{t+1})] \\ -q_t + \beta \cdot q_{t+1} \cdot (1 - \delta) \end{array} \right\} = 0 \quad (28)$$

$$\implies q_t = E_t \beta \cdot [F'(K_{t+1}) - C'(K_{t+1}) + q_{t+1} \cdot (1 - \delta)]. \quad (29)$$

⁵You will sometimes see the equation written

$$\frac{I_t}{K_t} = (q_t - 1),$$

which follows from the simplification $\bar{C} = 0$ and $b = 1$.

⁶In discrete time the timing of investment's effects on capital formation is somewhat arbitrary, as we have discussed earlier. One could have legitimately written $K_t - K_{t-1} = I_t - \delta K_{t-1}$. Perhaps it is a little more sensible to project K_{t+1} on I_t as I have done here under a "time to build" motivation of the transmission lag from investment outlays to capital deployed in production.

This is a *lovely* equation!⁷ It's an investment Euler equation. Note well it's interpretation. It says that the marginal value (shadow price) of a new unit of capital stock today (q_t) must equal the expected discounted value ($\beta = \frac{1}{(1+r)}$) of the the sum of three effects:

(i) the new unit's marginal contribution to real revenue, $F'(K_{t+1})$,

(ii) the contribution of the new unit to the marginal decline of future capital installation costs, $-C'K_{t+1}$,⁸

(iii) the depreciated shadow value of capital in the next period, $q_{t+1} \cdot (1 - \delta)$

If these conditions do not hold, the firm is missing a profitable intertemporal arbitrage opportunity.⁹

Now let's re-arrange terms in (29) to get q_{t+1} on the left-side:

$$q_t - E_t\beta \cdot q_{t+1} \cdot (1 - \delta) = E_t\beta \cdot [F'(K_{t+1}) - C'(K_{t+1})] \quad (30)$$

$$E_tq_t [1 - \beta \cdot (1 - \delta) L^{-1}] = E_t\beta \cdot [F'(K_{t+1}) - C'(K_{t+1})]. \quad (31)$$

Now solve for q_t :

$$q_t = \frac{E_t\beta \cdot [F'(K_{t+1}) - C'(K_{t+1})]}{[1 - \beta \cdot (1 - \delta) L^{-1}]}. \quad (32)$$

⁷It other derivations you may have seen the last term is sometimes appears without $(1 - \delta)$, because to keep the initial setup simple depreciation was assumed to be zero.

⁸Under the specification of costs proposed before,

$$C(I_t, K_t) = \frac{b}{2} \left(\frac{I_t}{K_t} - \bar{C} \right)^2 \cdot K_t,$$

the marginal adjustment cost is

$$C'(K_{t+1}) = \frac{b}{2} \left(\bar{C}^2 - \left(\frac{I_{t+1}}{K_{t+1}} \right)^2 \right).$$

Hence if $\left(\frac{I_{t+1}}{K_{t+1}} \right)^2 > \bar{C}^2$ (and \bar{C} is likely to be not far from zero), then a new unit of capital lowers future installation costs and, consequently, raises the expected PDV of net real revenue.

⁹See the discussion of perturbations to the Euler equation in the Consumption lecture notes. I could have gone through the same exercise here to nail the point.

Enumerating the forward polynomial

$$\frac{1}{[1 - \beta \cdot (1 - \delta) L^{-1}]} = \frac{1}{\left[1 - \left(\frac{1-\delta}{1+r}\right) L^{-1}\right]}$$

for (32) we obtain:

$$q_t = E_t \frac{1}{(1+r)} \sum_{j=0}^{\infty} \left(\frac{1-\delta}{1+r}\right)^j [F'(K_{t+1+j}) - C'(K_{t+1+j})]. \quad (33)$$

Again we have a *beautiful* equation, and you should understand well its interpretation. The equation says that the *marginal* value (shadow price) of an extra unit of capital (q) must equal the extra unit's *marginal* contribution to the expected PDV of real net revenues, where the operative discount factor incorporates the rate at which capital depreciates each period:

$$\frac{(1-\delta)}{(1+r)} < \left[\frac{1}{(1+r)} = \beta \right].$$

4 Tobin's q and Stock Prices

Hayashi (1982, *Econometrica*) proved that if the production technology and the adjustment cost function are homogeneous of degree 1 (CRS), as I have assumed here,¹⁰ then Tobin's *marginal* q equals *average* q , and average q gives the market value of the firm relative to the replacement cost of its capital stock.

¹⁰Note that the particular cost function proposed above

$$C(I_t, K_t) = \frac{b}{2} \left(\frac{I_t}{K_t} - \bar{C} \right)^2 \cdot K_t$$

is indeed homogenous of degree 1, that is,

$$C'(I) \cdot I + C'(K) \cdot K = C(I, K).$$

To see this, engage in the following exercise. I will take you through all the Mickey Mouse steps (most of them redundant for you) to obtain the result. First, multiply the solution in eq.(29) for q_t by K_{t+1} . This gives

$$K_{t+1} \cdot q_t = E_t \beta \cdot \left[\begin{array}{c} F'(K_{t+1}) \cdot K_{t+1} \\ -C'(K_{t+1}) \cdot K_{t+1} + q_{t+1} \cdot (1 - \delta) \cdot K_{t+1} \end{array} \right]. \quad (34)$$

By CRS we know that factor payments exhaust output

$$Q = F(K, A \cdot N) = F'(K) \cdot K + F'(N) \cdot N,$$

so the first right-side term in brackets in eq.(34) is

$$\begin{aligned} F'(K_{t+1}) \cdot K_{t+1} &= F(K_{t+1}, A \cdot N_{t+1}) - F'(N_{t+1}) \cdot N_{t+1} \\ &= F[K_{t+1}, A \cdot N_{t+1}] - w_{t+1} N_{t+1}. \end{aligned} \quad (35)$$

Hence eq.(34) may be expressed

$$K_{t+1} \cdot q_t = E_t \beta \cdot \left[\begin{array}{c} F(K_{t+1}, A \cdot N_{t+1}) - w_{t+1} N_{t+1} \\ -C'(K_{t+1}) \cdot K_{t+1} + q_{t+1} \cdot (1 - \delta) \cdot K_{t+1} \end{array} \right]. \quad (36)$$

From the capital accumulation equation evaluated at period $t+2$ we know that

$$K_{t+2} - K_{t+1} = I_{t+1} - \delta K_{t+1},$$

and so

$$(1 - \delta) \cdot K_{t+1} = K_{t+2} - I_{t+1}.$$

Therefore by definition

$$q_{t+1} \cdot (1 - \delta) \cdot K_{t+1} = q_{t+1} \cdot (K_{t+2} - I_{t+1}). \quad (37)$$

Now substitute $q_{t+1} \cdot (K_{t+2} - I_{t+1})$ for $q_{t+1} \cdot (1 - \delta) \cdot K_{t+1}$ in eq.(36):

$$\begin{aligned} K_{t+1} \cdot q_t &= E_t \beta \cdot \begin{bmatrix} F(K_{t+1}, A \cdot N_{t+1}) - w_{t+1} N_{t+1} \\ -C'(K_{t+1}) \cdot K_{t+1} + q_{t+1} \cdot (K_{t+2} - I_{t+1}) \end{bmatrix} \\ &= E_t \beta \cdot \begin{bmatrix} F(K_{t+1}, A \cdot N_{t+1}) - w_{t+1} N_{t+1} \\ -C'(K_{t+1}) \cdot K_{t+1} + q_{t+1} \cdot K_{t+2} - q_{t+1} \cdot I_{t+1} \end{bmatrix}. \end{aligned} \quad (38)$$

From the investment FOC we know that $q_{t+1} = 1 + C'(I_{t+1})$. Make this substitution for the last term in eq.(38):

$$\begin{aligned} K_{t+1} \cdot q_t &= E_t \beta \cdot \begin{bmatrix} F(K_{t+1}, A \cdot N_{t+1}) - w_{t+1} N_{t+1} \\ -C'(K_{t+1}) \cdot K_{t+1} + q_{t+1} \cdot K_{t+2} \\ -[1 + C'(I_{t+1})] \cdot I_{t+1} \end{bmatrix} \\ &= E_t \beta \cdot \begin{bmatrix} F(K_{t+1}, A \cdot N_{t+1}) - w_{t+1} N_{t+1} \\ -C'(K_{t+1}) \cdot K_{t+1} + q_{t+1} \cdot K_{t+2} \\ -C'(I_{t+1}) \cdot I_{t+1} - I_{t+1} \end{bmatrix}. \end{aligned} \quad (39)$$

Next remember our assumption (necessary for Hayashi's proof) that the adjustment cost function, like, the production function, is homogenous of degree 1,

$$C'(K_{t+1}) \cdot K_{t+1} + C'(I_{t+1}) \cdot I_{t+1} = C(I_{t+1}, K_{t+1}) \quad (40)$$

Make this substitution in eq.(39), giving

$$K_{t+1} \cdot q_t = E_t \beta \cdot \begin{bmatrix} F(K_{t+1}, A \cdot N_{t+1}) - w_{t+1} N_{t+1} \\ -C(I_{t+1}, K_{t+1}) + q_{t+1} \cdot K_{t+2} - I_{t+1} \end{bmatrix}. \quad (41)$$

We have obtained a difference equation for $K_{t+1} \cdot q_t$ with driving (forcing) function

$$E_t \beta \cdot \begin{bmatrix} F(K_{t+1}, A \cdot N_{t+1}) - w_{t+1} N_{t+1} \\ -C(I_{t+1}, K_{t+1}) - I_{t+1} \end{bmatrix}.$$

Note that the driving function is the net real revenue of the firm [eq.(18)]: Output less wage costs less the costs of installation of capital less direct capital investment costs. Use the usual methods to solve (41) by first moving $E_t [\beta q_{t+1} \cdot K_{t+2}]$ to the

left-side

$$E_t K_{t+1} \cdot q_t (1 - \beta L^{-1}) = E_t \beta \cdot \begin{bmatrix} F(K_{t+1}, A \cdot N_{t+1}) - w_{t+1} N_{t+1} \\ -C(I_{t+1}, K_{t+1}) - I_{t+1} \end{bmatrix} \quad (42)$$

and then divide through by and enumerate the forward polynomial operator $(1 - \beta L^{-1})$:

$$K_{t+1} \cdot q_t = E_t \beta \cdot \sum_{j=0}^{\infty} \beta^j \begin{bmatrix} F(K_{t+1+j}, A \cdot N_{t+1+j}) - w_{t+1+j} N_{t+1+j} \\ -C(I_{t+1+j}, K_{t+1+j}) - I_{t+1+j} \end{bmatrix}. \quad (43)$$

Next divide through by K_{t+1}

$$q_t = E_t \left\{ \frac{\beta \cdot \sum_{j=0}^{\infty} \beta^j \begin{bmatrix} F(K_{t+1+j}, A \cdot N_{t+1+j}) - w_{t+1+j} N_{t+1+j} \\ -C(I_{t+1+j}, K_{t+1+j}) - I_{t+1+j} \end{bmatrix}}{K_{t+1}} \right\}. \quad (44)$$

Finally we have arrived at yet another *gorgeous* equation! It says that q_t is the ratio of the expected present discounted value of the firm's net revenue flow to the market value of its capital stock. (Recall the price of capital was normed to 1.0). The numerator gives the 'fundamental' market value of a firm: The expected present discounted value of its net revenue flow, which is equal to its price per share times the number shares outstanding if asset markets are efficient. The denominator is the replacement cost of its capital stock. (Recall the real price of K is normed to 1.0). If producer's are investing optimally, the shadow price of an extra unit of investment (an extra unit of capital) should be equated to the firm's market value per unit of capital value.¹¹

¹¹Implicitly we observe the firm's market value 'ex-dividend', so that current payouts are already incorporated to current value, q_t . Note that we discount by β (the one-period discount factor) the expected PDV of time $t + 1$ net real revenues in order to put it on time t footing.

4.1 Investment and the Growth of the Capital Stock

Let's explore investment and growth of the capital stock a bit more. We know from the FOC for I_t (eq.26) that

$$\frac{I_t}{K_t} = \frac{1}{b} \cdot (q_t - 1) + \bar{C}$$

and from the dynamics of capital (eq.27) that

$$\frac{\Delta K_{t+1}}{K_t} = \frac{1}{b} \cdot (q_t - 1) + (\bar{C} - \delta)$$

The firm can acquire capital by building it, that is via investment, I , or it can buy capital stock on the stock market, by buying undervalued firms, or by buying back its own shares. When

$$q > 1 + b(\delta - \bar{C})$$

the firm is “expensive” relative to its fundamental value (or the entire stock market is expensive relative to its fundamental value). The optimal strategy is therefore to acquire capital stock by building it (eq.26):

$$\frac{I_t}{K_t} = \frac{1}{b} \cdot (q_t - 1) + \bar{C} > 0.$$

When

$$q < 1 + b(\delta - \bar{C}),$$

the firm is “cheap” (the market is “cheap”) and the optimal strategy is expend investment resources by buying capital stock on the stock market. In major contractions it is common to observe firms buying the “capital” of undervalued firms, rather building their own capital from scratch by conventional investment projects. Such arbitrage activities drive the ebb and flow of investment spending under q theory.

Return to the FOC for K_{t+1} (eq.33). We know

$$q_t = E_t \frac{1}{(1+r)} \sum_{j=0}^{\infty} \left(\frac{(1-\delta)}{(1+r)} \right)^j [F'(K_{t+1+j}) - C'(K_{t+1+j})]$$

Substituting into (26) we obtain

$$\frac{I_t}{K_t} = \frac{1}{b} \cdot \left(E_t \frac{1}{(1+r)} \sum_{j=0}^{\infty} \left(\frac{(1-\delta)}{(1+r)} \right)^j [F'(K_{t+1+j}) - C'(K_{t+1+j})] - 1 \right) + \bar{C} \quad (45)$$

The result is quite intuitive. The investment rate is a linear function of the shadow price, q . The shadow price is the expected PDV of marginal returns to capital. Since the capital stock grows at rate

$$\frac{\Delta K_{t+1}}{K_t} = \frac{I_t}{K_t} - \delta, \quad (46)$$

the dynamics of capital formation under q theory are:

$$\frac{\Delta K_{t+1}}{K_t} = \frac{1}{b} \cdot (q_t - 1) + (\bar{C} - \delta) \quad (47)$$

$$= \frac{1}{b} \cdot \left(E_t \frac{1}{(1+r)} \sum_{j=0}^{\infty} \left(\frac{(1-\delta)}{(1+r)} \right)^j \left[\begin{array}{c} F'(K_{t+1+j}) \\ -C'(K_{t+1+j}) \end{array} \right] - 1 \right) - \delta \quad (48)$$

Take $\bar{C} = \delta$. In this case we have

$$\begin{aligned} &> 0 \quad \text{at } q_t > 1 \\ \Delta K_{t+1} &= 0 \quad \text{at } q_t = 1 . \\ &< 0 \quad \text{at } q_t < 1 \end{aligned} \quad (49)$$

Contrast (48) to the static case discussed in the first part of the lecture in which capital changed without adjustment cost in one discrete jump to equate the time t marginal product of capital to its user cost.

4.2 The Dynamics of q and the Phase Diagram

To derive the dynamics of q we operate on eq.(30). This equation implies¹²

$$E_t q_{t+1} = \frac{(1+r)}{(1-\delta)} \cdot \{q_t - E_t [F'(K_{t+1}) - C'(K_{t+1})]\} \quad (50)$$

$$\begin{aligned} E_t (q_{t+1} - q_t) &= \frac{(r+\delta)}{(1-\delta)} \cdot q_t - E_t \frac{[F'(K_{t+1}) - C'(K_{t+1})]}{(1-\delta)} \\ &= \frac{(r+\delta)}{(1-\delta)} \cdot q_t - E_t \frac{f(k)}{(1-\delta)} \end{aligned} \quad (51)$$

where $f(k)$ is just short-hand for $[F'(K_{t+1}) - C'(K_{t+1})]$. Abstracting from expectations, note that

$$\begin{aligned} \Delta q_{t+1} &> 0 \quad \text{at } q_t > \frac{f(k)}{(r+\delta)} \\ \Delta q_{t+1} &= 0 \quad \text{at } q_t = \frac{f(k)}{(r+\delta)} \\ \Delta q_{t+1} &< 0 \quad \text{at } q_t < \frac{f(k)}{(r+\delta)} \end{aligned} \quad (52)$$

The dynamics of q in the q, K space follow. With CRS production, the locus of points or schedule along which $\Delta q_{t+1} = 0$ satisfies

$$q = \frac{f(k)}{(r+\delta)}, \quad (53)$$

which is *downward sloping* because

$$\frac{\partial q}{\partial f(k)} = \frac{F''(K) - C''(K)}{(r+\delta)} < 0, \quad (54)$$

given

$$F''(K) < 0 \quad (\text{diminishing returns}) \quad (55)$$

$$C''(K) > 0 \quad (\text{quadratic adjustment costs}). \quad (56)$$

¹²Eq.(50) is analogous to asset price relations, and is known as the arbitrage or no excess return relation. It has been argued that (50) implies that q and, hence, investment, approximately follows a martingale. For discussion see Blanchard, AER 1981.

So $q_t = [F'(K_{t+1}) - C'(K_{t+1})]$ unambiguously falls in K . ¹³[*Graph*]

With $\bar{C} = \delta$, the Phase diagram with unique equilibrium at $q = 1$, $K = K^*$ follows directly from the dynamics of K and q . [*Phase Graph*]

Lecture References:

Caballero, "Aggregate Investment," 1997

Blanchard and Fischer, Lecture on Macroeconomics, 1989, chapter 6

Hayashi, "Tobin's Marginal Q and Average Q," *Econometrica*, January 1982

Romer, *Advanced Macroeconomics*, 2006, chapter 8

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¹³For the specific functional form proposed for $C(I, K)$

$$C''(K) = b \left(\frac{I^2}{K^3} \right).$$