

Lag Operators and First-Order Difference Equations

1 A Couple of Series Results

A necessary condition for an infinite series

$$\sum_{j=0}^{\infty} c_j = c_0 + c_1 + c_2 + \dots + c_n + \dots$$

to converge is that

$$\lim_{n \rightarrow \infty} c_n = 0.$$

In other words, if the n th term of a series does not go to zero as $n \rightarrow \infty$, then the series diverges.

A convergent infinite series commonly appearing in economic models is a geometric series of the form $\sum_{j=0}^{\infty} \lambda^j$, where $|\lambda| < 1$. In this case

$$\sum_{j=0}^{\infty} \lambda^j \equiv S_{\infty} = \frac{1}{(1 - \lambda)}.$$

Proof:

Take the first n terms of S_{∞} ,

$$S_n \equiv \sum_{j=0}^{n-1} \lambda^j.$$

Now subtract λS_n

$$\begin{aligned} S_n - \lambda S_n &= (1 - \lambda) S_n \\ &= \lambda^0 + \lambda^1 + \lambda^2 + \dots + \lambda^{n-1} - \lambda^1 - \lambda^2 - \lambda^3 - \dots - \lambda^n \\ &= \lambda^0 - \lambda^n = (1 - \lambda^n). \end{aligned}$$

Hence we find,

$$(1 - \lambda) S_n = (1 - \lambda^n)$$

and therefore that

$$S_n = \frac{(1 - \lambda^n)}{(1 - \lambda)}$$

gives the partial sum of the first n terms (the finite sum of the geometric series). This result holds for any value of λ .

Further, when as postulated $|\lambda| < 1$, then $\lim_{n \rightarrow \infty} \lambda^n = 0$, and the series converges to

$$\lim_{n \rightarrow \infty} S_n \equiv S_\infty = \frac{1}{(1 - \lambda)}.$$

Note that the foregoing implies

$$\frac{1}{(1 - \lambda)} = 1 + \lambda + \lambda^2 + \lambda^3 + \dots + \lambda^\infty.$$

2 The Lag Operator

The lag-lead operator, L , is defined

$$\begin{aligned} L^0 X_t &= X_t, & L X_t &= X_{t-1}, \dots, L^n X_t = X_{t-n} \\ L^{-1} X_t &= X_{t+1}, \dots, L^{-n} X_t = X_{t+n} \end{aligned}$$

where n is any integer and $L^0 \equiv 1$.

Positive values of n define lags, negative values define leads. Note that the lag operator may be treated algebraically. For example:

$$\begin{aligned}
(1-L)(1-L)X_t &= \Delta X_t - \Delta X_{t-1} \\
&= (1-L)^2 X_t = (1-2L+L^2)X_t \\
&= X_t - 2X_{t-1} + X_{t-2}
\end{aligned}$$

and so forth.

2.1 Lag Algebra

Since the lag operator may be treated algebraically, the results above map onto it. For example:

$$\begin{aligned}
\sum_{j=0}^{\infty} (\lambda L)^j &= (\lambda^0 L^0 + \lambda^1 L^1 + \lambda^2 L^2 + \dots + \lambda^\infty L^\infty) \\
&= \frac{1}{(1-\lambda L)} \equiv (1-\lambda L)^{-1}, \quad |\lambda| < 1.
\end{aligned}$$

Note also that

$$(1-\lambda L)^{-1}(1-\lambda L) = 1.0$$

conforms to the expansion

$$\begin{aligned}
\sum_{j=0}^{\infty} (\lambda L)^j \cdot (1-\lambda L) &= \lambda^0 L^0 + \lambda^1 L^1 + \lambda^2 L^2 + \dots + \lambda^\infty L^\infty \\
&\quad - \lambda^1 L^1 - \lambda^2 L^2 - \dots - \lambda^\infty L^\infty \\
&= \lambda^0 L^0 = 1.0.
\end{aligned}$$

Finally, note that lag algebra applies in like manner to forward (lead) sequences:

$$\begin{aligned}
\sum_{j=0}^{\infty} (\lambda L)^{-j} &= \sum_{j=0}^{\infty} \left(\frac{1}{\lambda}\right)^j L^{-j} = 1 + \left(\frac{1}{\lambda}\right) L^{-1} + \left(\frac{1}{\lambda}\right)^2 L^{-2} + \left(\frac{1}{\lambda}\right)^3 L^{-3} + \dots \\
&= \frac{1}{\left(1 - \frac{1}{\lambda} L^{-1}\right)}, \quad |\lambda| < 1.
\end{aligned}$$

3 First-Order Dynamic Models with Lagged Input(s)

Consider the stable first-order model ($|\lambda| < 1$)

$$\begin{aligned} Y_t &= a + b \sum_{j=0}^{\infty} (\lambda L)^j X_t \\ &= a + bX_t + b\lambda X_{t-1} + b\lambda^2 X_{t-2} + b\lambda^3 X_{t-3} + \dots \end{aligned}$$

I shall assume throughout that $\{X_t\}_{t=-\infty}^{\infty}$ is a 'bounded sequence', that is, $|X_t| < \infty$ for all t . Given this assumption, if $|\lambda| < 1$, the outcome sequence $\{Y_t\}_{t=-\infty}^{\infty}$ produced by the model will be bounded and, as shown below, Y_t converges for X held fixed (the model is "stable").

From lag operator algebra we know that such first-order lag models may be written

$$Y_t = a + b \frac{X_t}{(1 - \lambda L)},$$

and after multiplying through by $(1 - \lambda L)$ we obtain

$$\begin{aligned} Y_t(1 - \lambda L) &= a(1 - \lambda) + bX_t \\ &= a(1 - \lambda) + \lambda Y_{t-1} + bX_t. \end{aligned}$$

Note that if the driving variable X is indefinitely held constant at some value, say \bar{X} , a solution to the above difference equation is

$$\begin{aligned} Y_t &= a + b \sum_{j=0}^{\infty} (\lambda L)^j \bar{X} \\ &= a + b \sum_{j=0}^{\infty} \lambda^j \bar{X} \\ &= a + b(\lambda^0 + \lambda^1 + \lambda^2 + \dots) \bar{X} \\ &= a + b \frac{\bar{X}}{(1 - \lambda)}. \end{aligned}$$

The model may readily be manipulated to show the response of Y to realizations of the driving variable(s) after some initial period, for example, after period $t = 0$.

Consider the model in the form $Y_t(1 - \lambda L) = a(1 - \lambda) + bX_t$, and multiply through by $(1 + \lambda^1 L^1 + \lambda^2 L^2 + \dots + \lambda^{t-1} L^{t-1})$, which gives:

$$\begin{aligned} & Y_t(1 - \lambda L) \cdot (1 + \lambda^1 L^1 + \lambda^2 L^2 + \dots + \lambda^{t-1} L^{t-1}) \\ = & a(1 - \lambda) \cdot (1 + \lambda^1 L^1 + \lambda^2 L^2 + \dots + \lambda^{t-1} L^{t-1}) \\ & + bX_t \cdot (1 + \lambda^1 L^1 + \lambda^2 L^2 + \dots + \lambda^{t-1} L^{t-1}), \end{aligned}$$

whence

$$\begin{aligned} & Y_t \cdot \left(\begin{array}{c} 1 + \lambda^1 L^1 + \lambda^2 L^2 + \dots + \lambda^{t-1} L^{t-1} \\ -\lambda^1 L^1 - \lambda^2 L^2 - \lambda^3 L^3 - \dots - \lambda^t L^t \end{array} \right) \\ = & a(1 - \lambda) \cdot (1 + \lambda^1 L^1 + \lambda^2 L^2 + \dots + \lambda^{t-1} L^{t-1}) \\ & + bX_t \cdot (1 + \lambda^1 L^1 + \lambda^2 L^2 + \dots + \lambda^{t-1} L^{t-1}). \end{aligned}$$

Recalling the result for partial sums, $\sum_{j=0}^{t-1} (\lambda L)^j = \frac{(1 - (\lambda L)^t)}{(1 - \lambda L)}$, this exercise yields

$$Y_t \cdot (1 - \lambda^t L^t) = a(1 - \lambda^t) + \sum_{j=0}^{t-1} (\lambda L)^j X_t,$$

and therefore

$$Y_t = a(1 - \lambda^t) + \lambda^t Y_0 + b \sum_{j=0}^{t-1} (\lambda L)^j X_t,$$

or

$$\begin{aligned} Y_t &= a + \lambda^t (Y_0 - a) + b \sum_{j=0}^{t-1} (\lambda L)^j X_t \\ &= a + \lambda^t (Y_0 - a) + b \sum_{j=0}^{t-1} \lambda^j X_{t-j}, \\ t &\succeq 1, \end{aligned}$$

where Y_0 is the initial condition (starting value) of Y .

If the driving variable(s) of this difference equation had been held fixed at \bar{X} from period $t = 1$ to period $t = t$, the foregoing results imply the solution

$$Y_t = a + \lambda^t (Y_0 - a) + b \frac{(1 - \lambda^t)}{(1 - \lambda)} \bar{X}.$$

Note that the same result may be obtained more tediously by repeated substitution for lagged Y 's.

3.1 Solving Lag Models Forward

Recall that in order for the dynamic lag model $Y_t = a + b \frac{X_t}{(1 - \lambda L)}$ to yield a bounded sequence $\{Y_t\}_{t=-\infty}^{\infty}$ which converges (does not "explode") when driving the variable(s) are held at constant value(s), it is necessary that $|\lambda| < 1$.

A non-stable first-order difference equation where $|\lambda| > 1$ may be solved forward in time to achieve a stable representation, and conversely. (Forward models often arise naturally in rational expectations setups.)

To see this, take the "forward inverse" of the lag model by multiplying the geometric polynomial term of the driving variable by $\frac{-(\lambda L)^{-1}}{-\lambda L^{-1}} (= 1.0)$. This exercise evidently shows that an explosive, first-order lag model can be expressed as a stable, first-order forward model, that is, in terms of future realizations of X from period $t + 1$:

$$\begin{aligned} Y_t &= a + b \frac{X_t}{(1 - \lambda L)} = a + b \frac{X_t}{(1 - \lambda L)} \cdot \begin{bmatrix} -(\lambda L)^{-1} \\ -(\lambda L)^{-1} \end{bmatrix} \\ &= a + b \frac{-(\lambda L)^{-1}}{[1 - (\lambda L)^{-1}]} X_t \\ &= a - b \left(\frac{1}{\lambda} \right) \frac{1}{[1 - (\frac{1}{\lambda}) L^{-1}]} X_{t+1} \\ &= a - b \left(\frac{1}{\lambda} \right) \sum_{j=0}^{\infty} \left(\frac{1}{\lambda} \right)^j L^{-j} X_{t+1} \\ &= a - b \left(\frac{1}{\lambda} \right) \sum_{j=0}^{\infty} \left(\frac{1}{\lambda} \right)^j X_{t+1+j} \end{aligned}$$

or

$$Y_t = a - b \sum_{j=1}^{\infty} \left(\frac{1}{\lambda}\right)^j X_{t+j}.$$

Given $\left|\frac{1}{\lambda}\right| < 1$, that the sequence $\{Y_t\}_{t=-\infty}^{\infty}$ is bounded when the input sequence $\{X_t\}_{t=-\infty}^{\infty}$ is bounded, as already noted. Using the representation

$$Y_t = a - b \left(\frac{1}{\lambda}\right) \sum_{j=0}^{\infty} \left(\frac{1}{\lambda}\right)^j L^{-j} X_{t+1},$$

we see that for future realizations of X held at some fixed value \bar{X} , Y converges to

$$\begin{aligned} Y_t &= a - b \left(\frac{1}{\lambda}\right) \cdot \left[\frac{1}{\left(1 - \left(\frac{1}{\lambda}\right)\right)} \right] \cdot \bar{X} \\ &= a - \left[\frac{b}{(\lambda - 1)} \right] \cdot \bar{X}. \end{aligned}$$

The first-order lag model may also be solved as a finite sum forward projection. Use the forward representation

$$Y_t \cdot [1 - (\lambda L)^{-1}] = a \cdot [1 - (\lambda L)^{-1}] - b(\lambda L)^{-1} X_t$$

and multiply left and right sides by

$$\left[1 + (\lambda L)^{-1} + (\lambda L)^{-2} + \dots + (\lambda L)^{-(T-1)}\right], \quad T > 0.$$

After so doing we obtain

$$\begin{aligned} & Y_t \cdot [1 - (\lambda L)^{-T}] \\ &= a \cdot [1 - (\lambda L)^{-T}] - b \left[(\lambda L)^{-1} + (\lambda L)^{-2} + \dots + (\lambda L)^{-T} \right] X_t \end{aligned}$$

Moving the lagged Y term to the right-side gives

$$\begin{aligned}
Y_t &= a \left[1 - \left(\frac{1}{\lambda} \right)^T \right] + \left(\frac{1}{\lambda} \right)^T Y_{t+T} - b \sum_{j=1}^T \left(\frac{1}{\lambda} \right)^j X_{t+j} \\
&= a \left[1 - \left(\frac{1}{\lambda} \right)^T \right] + \left(\frac{1}{\lambda} \right)^T Y_{t+T} - b \left(\frac{1}{\lambda} \right) \sum_{j=0}^{T-1} \left(\frac{1}{\lambda} \right)^j X_{t+j+1}.
\end{aligned}$$

You may verify the results above "manually" by repeated forward substitution. Note that for $T = 1$, we obtain the equation

$$Y_t = a \left[1 - \left(\frac{1}{\lambda} \right) \right] + \left(\frac{1}{\lambda} \right) Y_{t+1} - b \left(\frac{1}{\lambda} \right) X_{t+1}$$

which is a form that often arises in rational expectations models, where the right-side one-period-ahead terms are time t expectations $E_t Y_{t+1}$ and $E_t X_{t+1}$.

Finally, note that one can go from a forward solution to a backward-lag solution by applying the "backward inverse" $\frac{-(\lambda L^{-1})^{-1}}{-\lambda L^{-1} - 1}$ ($= 1.0$) to the model's forward geometric polynomial term, with boundedness and convergence dependent, as noted already, on the absolute value of λ and the boundedness of $\{X_t\}_{t=-\infty}^{\infty}$.

4 A Simple Example: A 'Market Fundamentals' Model of Asset Prices

Let P be the asset price (say a stock price), and D be the dividend.

Assume the following sequence of action and dividend payment time line: An investor buys the asset at time t at price P_t *after* any period t dividend (D_t) has been paid, and the investor sells at period $t+1$ at price P_{t+1} *after* any period $t+1$ dividend (D_{t+1}) has been paid ("ex-dividend" prices). I shall assume for present expositional purposes that the investor's expected return is constant, $E_t r_{t+1} = r > 0$ for all t .

The expected rate of return over any period equals the expected capital gain plus the expected dividend rate

$$r = E_t \left[\frac{(P_{t+1} - P_t) + D_{t+1}}{P_t} \right]$$

where E_t is the expectation conditioned on knowledge of outcomes up to and including period t (the period t information set).

Now solve for the asset purchase price, P_t , noting that under the assumed information set $E_t P_t = P_t$, and that the lead operator is restricted so that $L^{-1} E_t X_t = E_t X_{t+1}$.

$$r P_t = E_t (P_{t+1} - P_t + D_{t+1})$$

$$r P_t + P_t = E_t (P_{t+1} + D_{t+1})$$

$$(1 + r) P_t = E_t (P_{t+1} + D_{t+1})$$

$$P_t = E_t \left[\left(\frac{1}{R} \right) \cdot (P_{t+1} + D_{t+1}) \right],$$

where $R \equiv (1 + r)$.

Hence

$$\begin{aligned} P_t - \left[E_t \left(\frac{1}{R} \right) P_{t+1} \right] &= E_t \left[\left(\frac{1}{R} \right) D_{t+1} \right] \\ E_t P_t \cdot \left[1 - \left(\frac{1}{R} \right) L^{-1} \right] &= E_t \left[\left(\frac{1}{R} \right) D_{t+1} \right], \end{aligned}$$

and therefore

$$\begin{aligned} P_t &= E_t \frac{\left[\left(\frac{1}{R} \right) D_{t+1} \right]}{\left[1 - \left(\frac{1}{R} \right) L^{-1} \right]} \\ &= E_t \left[\left(\frac{1}{R} \right) \sum_{j=0}^{\infty} \left(\frac{1}{R} \right)^j L^{-j} D_{t+1} \right] \\ &= E_t \left[\sum_{j=1}^{\infty} \left(\frac{1}{R} \right)^j D_{t+j} \right] \end{aligned}$$

which is a "market fundamentals" model of asset price, with price being determined by the Present Discounted Value of the expected dividend payment stream. This result may be verified by repeated substitution, keeping in mind the law of iterated expectations,

$$E_t[E_{t+j}(X)] = E_t(X), \quad j \succeq 0,$$

and the restriction that the conditional expectation operator dated at the evaluation period t is not affected by the lead operator: $L^{-1}E_tX_t = E_tX_{t+1} \neq E_{t+1}X_{t+1}$ and so forth. [To pin down clearly this restriction on the lead operator, a separate expectations lead operator is sometimes introduced; see, for example, Sargent (1987) p.395 who proposes the operator B such that $B^{-n}E_tX_{t+j} = E_tX_{t+j+n}$, reserving L to function such that $L^{-n}E_tX_{t+j} = E_{t+n}X_{t+j+n}$. I give L double duty here by just imposing the side restriction that it does not shift in time the conditional expectation operator E_t .]

Note that for expected dividends held constant at say, \bar{D} , the asset price converges, since $\frac{1}{R} \equiv \frac{1}{(1+r)}$ is less than 1.0 given $r > 0$. Using the representation

$$P_t = E_t \left[\left(\frac{1}{R} \right) \sum_{j=0}^{\infty} \left(\frac{1}{R} \right)^j L^{-j} D_{t+1} \right]$$

when dividends are fixed at \bar{D} , we find

$$\begin{aligned} P_t &= \left[\frac{1}{\left(1 - \left(\frac{1}{R}\right)\right)} - 1 \right] \bar{D} \\ &= \frac{1}{(R - 1)} \bar{D} \\ &= \left(\frac{1}{r} \right) \bar{D}. \end{aligned}$$

Lecture References:

J.Y. Campbell, A.W. Lo and A.C. MacKinlay, *The Econometrics of Financial Markets* (1997), chapter 7.

J.D. Hamilton, *Time Series Analysis* (1994), chapter 2.

T. Sargent, *Macroeconomic Theory* (1987), chapter 9.