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# Lag Operators and First-Order Difference Equations

## 1 A Couple of Series Results

A necessary condition for an infinite series

$$\sum_{j=0}^{\infty} cj = c0 + c1 + c2 + ... + cn + ...$$

to converge is that

$$\lim_{n\to\infty} cn = 0.$$

In other words, if the nth term of a series does not go to zero as $n \to \infty$, then the series diverges.

A convergent infinite series commonly appearing in economic models is a geometric series of the form $\sum_{j=0}^{\infty} \lambda^j$, where $|\lambda| < 1$. In this case

$$\sum_{j=0}^{\infty} \lambda^j \equiv S_\infty = \frac{1}{(1-\lambda)}.$$

*Proof*:

Consider the first $n$ terms of $S_\infty$

$$S_n \equiv \sum_{j=0}^{n-1} \lambda^j$$

where recall that $\lambda^0 = 1$.

Now subtract $\lambda S_n$

$$\begin{aligned} S_n - \lambda S_n &= (1-\lambda) S_n \\ &= \lambda^0 + \lambda^1 + \lambda^2 + ... + \lambda^{n-1} - \lambda^1 - \lambda^2 - \lambda^3 - ... - \lambda^n \\ &= \lambda^0 - \lambda^n = (1 - \lambda^n). \end{aligned}$$

Hence we find

$$(1-\lambda) S_n = (1-\lambda^n)$$

and therefore that

$$S_n = \frac{(1-\lambda^n)}{(1-\lambda)}$$

which gives the partial sum of the first $n$ terms (the finite sum of the geometric series). This partial sum result holds for *any value* of $\lambda$.

Further, when as postulated $|\lambda| < 1$, then $\lim_{n\to\infty} \lambda^n = 0$, and the series converges to

$$\lim_{n\to\infty} S_n \equiv S_\infty = \frac{1}{(1-\lambda)}.$$

Note that the foregoing implies

$$\frac{1}{(1-\lambda)} = 1 + \lambda + \lambda^2 + \lambda^3 + ... + \lambda^\infty.$$

# 2 The Lag Operator

The lag-lead operator, $L$, is defined

$$L^0 X_t = X_t, \qquad L X_t = X_{t-1}, ... L^n X_t = X_{t-n}$$
$$L^{-1} X_t = X_{t+1}, ... L^{-n} X_t = X_{t+n}$$

where $n$ is any integer and $L^0 \equiv 1$.

Positive values of $n$ define lags, negative values define leads. Note that the lag operator may be treated algebraically. For example:

$$\begin{aligned}(1-L)(1-L)X_t &= \Delta X_t - \Delta X_{t-1} \\ &= (1-L)^2 X_t = \left(1-2L+L^2\right) X_t \\ &= X_t - 2X_{t-1} + X_{t-2}\end{aligned}$$

and so forth.

## 2.1 Lag Algebra

Since the lag operator may be treated algebraically, the results above map onto it. For example:

$$\begin{aligned}\sum_{j=0}^{\infty}(\lambda L)^j &= \left(\lambda^0 L^0 + \lambda^1 L^1 + \lambda^2 L^2 + ... + \lambda^\infty L^\infty\right) \\ &= \frac{1}{(1-\lambda L)} \equiv (1-\lambda L)^{-1}.\end{aligned}$$

Note also that

$$(1-\lambda L)^{-1}(1-\lambda L) = 1.0$$

conforms to the expansion

$$\begin{aligned}\sum_{j=0}^{\infty}(\lambda L)^j \cdot (1-\lambda L) &= \lambda^0 L^0 + \lambda^1 L^1 + \lambda^2 L^2 + ... + \lambda^\infty L^\infty \\ &\quad -\lambda^1 L^1 - \lambda^2 L^2 - ... - \lambda^\infty L^\infty \\ &= \lambda^0 L^0 = 1.0.\end{aligned}$$

Finally, note that lag algebra applies in like manner to forward (lead) sequences:

$$\sum_{j=0}^{\infty} (\lambda L)^{-j} = \sum_{j=0}^{\infty} \left(\frac{1}{\lambda}\right)^{j} L^{-j} = 1 + \left(\frac{1}{\lambda}\right) L^{-1} + \left(\frac{1}{\lambda}\right)^{2} L^{-2} + \left(\frac{1}{\lambda}\right)^{3} L^{-3} + ...$$

$$= \frac{1}{\left(1 - \frac{1}{\lambda} L^{-1}\right)}.$$

# 3 First-Order Deterministic Dynamic Models with Exogenous Input(s)

Consider the first-order model

$$\begin{aligned} Y_t &= a + b \sum_{j=0}^{\infty} (\lambda L)^{j} X_t \\ &= a + bX_t + b\lambda X_{t-1} + b\lambda^2 X_{t-2} + b\lambda^3 X_{t-3} + ... \\ &= a + b \frac{X_t}{(1 - \lambda L)}. \end{aligned}$$

After multiplying through by $(1 - \lambda L)$ we obtain

$$\begin{aligned} Y_t (1 - \lambda L) &= a (1 - \lambda) + bX_t \\ &\Rightarrow Y_t = a (1 - \lambda) + \lambda Y_{t-1} + bX_t \end{aligned}$$

a result that may be confirmed manually by repeated substitution of lagged $Y's$. Note that the above expressions hold for any value of the lag weighting parameter $\lambda$.

I shall assume throughout that $\{X_t\}_{t=-\infty}^{\infty}$ is a 'bounded sequence', that is, $|X_t| < \infty$ for all $t$. Given that assumption, if $|\lambda| < 1$ the outcome sequence $\{Y_t\}_{t=-\infty}^{\infty}$ generated by the model will be bounded.

If the driving variable $X$ is indefinitely held constant at some value, say $\overline{X}$, and if $|\lambda| < 1$, the above first-order equation converges, as implied by the earlier results on infinite geometric series:

$$\begin{aligned} Y_t &= a + b\sum_{j=0}^{\infty} (\lambda L)^j \, \overline{X} \\ &= a + b\sum_{j=0}^{\infty} \lambda^j \overline{X} \\ &= a + b\left(\lambda^0 + \lambda^1 + \lambda^2 + ...\right) \overline{X} \\ &\Rightarrow \; \overline{Y} = a + b\frac{\overline{X}}{(1-\lambda)} \end{aligned}$$

where $\overline{Y}$ is the convergent, steady-state value of $Y$.

The model may readily be manipulated to show the response of $Y$ to realizations of the driving variable(s) after some initial period, for example, after period $t = 0$. Consider the model in the form

$$Y_t \left(1 - \lambda L\right) = a\left(1 - \lambda\right) + bX_t$$

and multiply through by

$$\left(1 + \lambda^1 L^1 + \lambda^2 L^2 + ... + \lambda^{t-1} L^{t-1}\right)$$

which gives:

$$\begin{aligned} & Y_t \left(1 - \lambda L\right) \cdot \left(1 + \lambda^1 L^1 + \lambda^2 L^2 + ... + \lambda^{t-1} L^{t-1}\right) \\ = \; & a\left(1 - \lambda\right) \cdot \left(1 + \lambda^1 L^1 + \lambda^2 L^2 + ... + \lambda^{t-1} L^{t-1}\right) \\ & + bX_t \cdot \left(1 + \lambda^1 L^1 + \lambda^2 L^2 + ... + \lambda^{t-1} L^{t-1}\right), \end{aligned}$$

whence

$$
\begin{aligned}
& Y_t \cdot \begin{pmatrix} 1+\lambda^1 L^1+\lambda^2 L^2+\ldots+\lambda^{t-1} L^{t-1} \\ -\lambda^1 L^1-\lambda^2 L^2-\lambda^3 L^3-\ldots-\lambda^t L^t \end{pmatrix} \\
= \; & a(1-\lambda) \cdot \left(1+\lambda^1 L^1+\lambda^2 L^2+\ldots+\lambda^{t-1} L^{t-1}\right) \\
& +b X_t \cdot \left(1+\lambda^1 L^1+\lambda^2 L^2+\ldots+\lambda^{t-1} L^{t-1}\right).
\end{aligned}
$$

Recalling the result for partial sums, $\sum_{j=0}^{t-1}(\lambda L)^j=\frac{\left(1-(\lambda L)^t\right)}{(1-\lambda L)}$, the expression above simplifies to

$$
Y_t \cdot\left(1-\lambda^t L^t\right)=a\left(1-\lambda^t\right)+b \sum_{j=0}^{t-1}(\lambda L)^j X_t,
$$

and therefore

$$
Y_t=a\left(1-\lambda^t\right)+\lambda^t Y_0+b \sum_{j=0}^{t-1}(\lambda L)^j X_t,
$$

or

$$
\begin{aligned}
Y_t & = a+\lambda^t\left(Y_0-a\right)+b \sum_{j=0}^{t-1}(\lambda L)^j X_t \\
& = a+\lambda^t\left(Y_0-a\right)+b \sum_{j=0}^{t-1} \lambda^j X_{t-j}, \\
\text{for } t & \succeq 1 \text{ and any value of } \lambda
\end{aligned}
$$

where $Y_0$ is the initial condition (starting value) of $Y$.

If the driving variable(s) of the foregoing difference equation had been held fixed at some value $X_t=\overline{X}$ from period $t=1$ to period $t=t$, the foregoing results imply the solution

$$
Y_t=a+\lambda^t\left(Y_0-a\right)+b \frac{\left(1-\lambda^t\right)}{(1-\lambda)} \overline{X}
$$

which again holds for any value of $\lambda$. As noted earlier, this result may be obtained more tediously by repeated substitution for lagged $Y's$.

## 3.1 Solving Lag Models Forward

Recall that in order for the first-order dynamic lag model $Y_t = a + b\frac{X_t}{(1-\lambda L)}$ to yield a bounded sequence $\{Y_t\}_{t=-\infty}^{\infty}$ which converges (does not "explode") when driving the variable(s) are held at constant value(s), it is necessary that $|\lambda| < 1$.

A non-stable first-order difference equation in which $|\lambda| > 1$ may be solved forward in time to achieve a stable representation, and conversely. (Forward models often arise naturally in rational expectations setups.)

To see this, take the "forward inverse" of the lag model by multiplying the geometric polynomial term of the driving variable by $\frac{-(\lambda L)^{-1}}{-(\lambda L)^{-1}}(= 1.0)$. This exercise evidently shows that an explosive, first-order lag model ($\lambda > 1$) can be expressed as a stable, first-order forward model, that is, in terms of future realizations of $X_t$ from period $t + 1$:

$$
\begin{aligned}
Y_t &= a + b\frac{X_t}{(1-\lambda L)} = a + b\frac{X_t}{(1-\lambda L)} \cdot \left[\frac{-(\lambda L)^{-1}}{-(\lambda L)^{-1}}\right] \\
&= a + b\frac{-(\lambda L)^{-1}}{\left[1-(\lambda L)^{-1}\right]} X_t \\
&= a - b\left(\frac{1}{\lambda}\right)\frac{1}{\left[1-\left(\frac{1}{\lambda}\right)L^{-1}\right]} X_{t+1} \\
&= a - b\left(\frac{1}{\lambda}\right)\sum_{j=0}^{\infty}\left(\frac{1}{\lambda}\right)^{j} L^{-j} X_{t+1} \\
&= a - b\left(\frac{1}{\lambda}\right)\sum_{j=0}^{\infty}\left(\frac{1}{\lambda}\right)^{j} X_{t+1+j} \\
&\quad \text{or} \\
Y_t &= a - b\sum_{j=1}^{\infty}\left(\frac{1}{\lambda}\right)^{j} X_{t+j} \\
&= a - b\cdot\left\{\left(\frac{1}{\lambda}\right)X_{t+1} + \left(\frac{1}{\lambda}\right)^{2}X_{t+2} + \left(\frac{1}{\lambda}\right)^{3}X_{t+3} + ... + \left(\frac{1}{\lambda}\right)^{\infty}X_{t+\infty}\right\}.
\end{aligned}
$$

Given $\left|\frac{1}{\lambda}\right| < 1$, the sequence $\{Y_t\}_{t=-\infty}^{\infty}$ is bounded when the input sequence $\{X_t\}_{t=-\infty}^{\infty}$ is bounded, as already noted.

Using the representation

$$
Y_t = a - b\left(\frac{1}{\lambda}\right)\sum_{j=0}^{\infty}\left(\frac{1}{\lambda}\right)^{j} L^{-j} X_{t+1},
$$

we see that when when $|\frac{1}{\lambda}| < 1$ and future realizations of $X_t$ are held at some fixed value $\overline{X}$, $Y_t$ eventually converges to the steady-state value

$$
\begin{aligned}
Y_t \;\Rightarrow\; \overline{Y} = a - b\left(\frac{1}{\lambda}\right) \cdot \left[\frac{1}{(1-(\frac{1}{\lambda}))}\right] \cdot \overline{X} \\
= \; a - \left[\frac{b}{(\lambda - 1)}\right] \cdot \overline{X}.
\end{aligned}
$$

Note that the first-order lag model may also be solved as a finite sum forward projection. Multiplying both sides of the preceding forward model

$$
\begin{aligned}
Y_t &= a - b\left(\frac{1}{\lambda}\right) \sum_{j=0}^{\infty} \left(\frac{1}{\lambda}\right)^j L^{-j} X_{t+1} \\
&= a - b\left(\frac{1}{\lambda}\right) \sum_{j=0}^{\infty} (\lambda L)^{-j} X_{t+1} \\
&= a - b\left(\frac{1}{\lambda L}\right) \sum_{j=0}^{\infty} (\lambda L)^{-j} X_t \\
&= a - b\,(\lambda L)^{-1} \cdot \frac{1}{\left[1 - (\lambda L)^{-1}\right]} X_t
\end{aligned}
$$

by $\left[1 - (\lambda L)^{-1}\right]$ we obtain the forward representation

$$
Y_t \cdot \left[1 - (\lambda L)^{-1}\right] = a \cdot \left[1 - (\lambda L)^{-1}\right] - b\,(\lambda L)^{-1} X_t
$$

Now for $T > 0$ multiply left- and right-sides by

$$
\left[1 + (\lambda L)^{-1} + (\lambda L)^{-2} + ... + (\lambda L)^{-(T-1)}\right] = \frac{\left[1 - (\lambda L)^{-T}\right]}{\left[1 - (\lambda L)^{-1}\right]}
$$

yielding

$$
\begin{aligned}
Y_t \cdot \left[1-(\lambda L)^{-1}\right] \cdot \frac{\left[1-(\lambda L)^{-T}\right]}{\left[1-(\lambda L)^{-1}\right]} &= a \cdot \left[1-(\lambda L)^{-1}\right] \cdot \frac{\left[1-(\lambda L)^{-T}\right]}{\left[1-(\lambda L)^{-1}\right]} \\
&\quad -b\,(\lambda L)^{-1} X_t \cdot \left[\begin{array}{c} 1+(\lambda L)^{-1}+(\lambda L)^{-2} \\ +...+(\lambda L)^{-(T-1)} \end{array}\right] \\
Y_t \cdot \left[1-(\lambda L)^{-T}\right] &= a \cdot \left[1-\lambda^{-T}\right] \\
&\quad -b\left[(\lambda L)^{-1}+(\lambda L)^{-2}+...+(\lambda L)^{-T}\right] \cdot X_t
\end{aligned}
$$

Moving the term $-(\lambda L)^{-T} \cdot Y_t$ on the left-side of the equation above to the right-side, gives

$$
\begin{aligned}
Y_t &= a\left[1-\left(\frac{1}{\lambda}\right)^T\right] + \left(\frac{1}{\lambda}\right)^T Y_{t+T} - b\sum_{j=1}^{T}\left(\frac{1}{\lambda}\right)^j X_{t+j} \\
&= a\left[1-\left(\frac{1}{\lambda}\right)^T\right] + \left(\frac{1}{\lambda}\right)^T Y_{t+T} - b\left(\frac{1}{\lambda}\right)\sum_{j=0}^{T-1}\left(\frac{1}{\lambda}\right)^j X_{t+j+1}.
\end{aligned}
$$

You may verify the previous results "manually" by repeated forward substitution. Note that for $T = 1$, we obtain the first-order lead representation

$$
Y_t = a\left[1-\left(\frac{1}{\lambda}\right)\right] + \left(\frac{1}{\lambda}\right) Y_{t+1} - b\left(\frac{1}{\lambda}\right) X_{t+1}
$$

which is a form that often arises in rational expectations models, where the right-side one-period-ahead terms are time $t$ expectations of $Y_{t+1}$ and $X_{t+1}$ : $E_t Y_{t+1}$ and $E_t X_{t+1}$.

Note that one can go from a forward-lead solution to a backward-lag solution by applying the "backward inverse" $\frac{-\left(\lambda L^{-1}\right)^{-1}}{-\left(\lambda L^{-1}\right)^{-1}}$ $(= 1.0)$ to the model's forward geometric polynomial term, with boundedness and convergence dependent, as noted already, on the absolute value of $\lambda$, $\frac{1}{\lambda}$ and the boundedness of $\{X_t\}_{t=-\infty}^{\infty}$. Finally, the basic principles laid out above apply to higher order difference equations/dynamic models. I leave those extensions as exercises for the ambitious.

# 4 A Simple Example: A 'Market Fundamentals' Model of Asset Prices

Let $P$ be the asset price (say a stock price), and $D$ be the dividend.

Assume the following sequence of action and dividend payment time line: An investor buys the asset at time $t$ at price $P_t$ *after* any period $t$ dividend ($D_t$) has been paid, and the investor sells at period $t+1$ at price $P_{t+1}$ *after* any period $t+1$ dividend ($D_{t+1}$) has been paid ("ex-dividend" prices). I shall assume for present expositional purposes that the investor's expected return is constant, $E_t r_{t+1} = r > 0$ for all $t$.

The (in this simple example, constant) expected rate of return over any period equals the expected capital gain plus the expected dividend payout rate

$$r = E_t \left[ \frac{(P_{t+1} - P_t) + D_{t+1}}{P_t} \right]$$

where $E_t$ is the expectation conditioned on knowledge of outcomes up to and including period $t$ (the period $t$ information set).

Now solve for the asset purchase price, $P_t$, noting that under the assumed information set $E_t P_t = P_t$, and that the lead operator is restricted so that $L^{-1} E_t X_t = E_t X_{t+1}$ (in other words the lead operator $L^n$, $n = 1, 2, ...$) does not affect the time $t$ expectation operator $E_t$.

$$rP_t = E_t \left( P_{t+1} - P_t + D_{t+1} \right)$$

$$rP_t + P_t = E_t \left( P_{t+1} + D_{t+1} \right)$$

$$(1 + r) P_t = E_t \left( P_{t+1} + D_{t+1} \right)$$

$$P_t = E_t \left[ \left( \frac{1}{R} \right) \cdot \left( P_{t+1} + D_{t+1} \right) \right],$$

where $R \equiv (1+r)$.

Hence

$$P_t - \left[E_t\left(\frac{1}{R}\right)P_{t+1}\right] = E_t\left[\left(\frac{1}{R}\right)D_{t+1}\right]$$

$$E_tP_t \cdot \left[1 - \left(\frac{1}{R}\right)L^{-1}\right] = E_t\left[\left(\frac{1}{R}\right)D_{t+1}\right],$$

and therefore

$$\begin{aligned} P_t &= E_t\frac{\left[\left(\frac{1}{R}\right)D_{t+1}\right]}{\left[1-\left(\frac{1}{R}\right)L^{-1}\right]} \\ &= E_t\left[\left(\frac{1}{R}\right)\sum_{j=0}^{\infty}\left(\frac{1}{R}\right)^j L^{-j}D_{t+1}\right] \\ &= E_t\left[\sum_{j=1}^{\infty}\left(\frac{1}{R}\right)^j D_{t+j}\right] \end{aligned}$$

which is a overly simplified "market fundamentals" model of asset price, with price being determined by the Present Discounted Value of the expected dividend payment stream, the discount rate being $\frac{1}{R} \equiv \frac{1}{(1+r)}$. This result may be verified by repeated substitution, keeping in mind the law of iterated expectations,

$$E_t[E_{t+j}(X)] = E_t(X), \quad j \succeq 0,$$

and the restriction mentioned earlier that the conditional expectation operator dated at the evaluation period $t$ is not affected by the lead operator: $L^{-1}E_tX_t = E_tX_{+1} \neq E_{t+1}X_{t+1}$ and so forth. [To pin down clearly this restriction on the lead operator, a separate expectations lead operator is sometimes introduced;

see, for example, Sargent (1987) p.395 who proposes the operator $B$ such that $B^{-n}E_tX_{t+j} = E_tX_{t+j+n}$, reserving $L$ to function such that $L^{-n}E_tX_{t+j} = E_{t+n}X_{t+j+n}$. Sargent's proposal never gained traction and along with others I give $L$ double duty here by just imposing the side restriction that it does not shift in time the conditional expectation operator $E_t$.]

Note that for expected dividends held constant at say, $\overline{D}$, the asset price converges, since $\frac{1}{R} \equiv \frac{1}{(1+r)}$ is less than 1.0 given $r > 0$. Using the representation

$$P_t = E_t\left[\left(\frac{1}{R}\right)\sum_{j=0}^{\infty}\left(\frac{1}{R}\right)^j L^{-j}D_{t+1}\right]$$

when dividends are fixed at $\overline{D}$, we find

$$\begin{aligned} P_t &= \left[\frac{1}{(1-(\frac{1}{R}))} - 1\right]\overline{D} \\ &= \frac{1}{(R-1)}\overline{D} \\ &= \left(\frac{1}{r}\right)\overline{D}. \end{aligned}$$

**Lecture References**:

J.Y. Campbell, A.W. Lo and A.C. MacKinlay, *The Econometrics of Financial Markets* (1997), chapter 7.

J.D. Hamilton, *Time Series Analysis* (1994), chapter 2.

T. Sargent, *Macroeconomic Theory* (1987), chapter 9.