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Overlapping Generations

1 Households

People have two period economic lives and do not care about future generations. Hence there are no bequests, no dynastic behavior, no altruism. In period 1 agents work and save (the "young"). In period 2, they consume and then die (the "old"). At every period there is an overlapping generation of one young and one old cohort of agents. A "period" is a generation, and so should be thought of as around 35-40 years of annual time. Everything is real and expressed in units of consumables.

1.1 Utility

The Utility maximand is analogous to that used in my Ramsey-Cass-Koopmans lectures, except it has a finite, two period horizon and time is discrete:

$$U_t = u\left[c_{1,t}\right] + \frac{1}{(1+\rho)} \cdot u\left[c_{2,t+1}\right] \tag{1}$$

where $\rho > 0$, $c_{1,t}$ is the consumption of generation t when young (in period t) and $c_{2,t+1}$ is the consumption of generation t when old (in period $t+1$).

Utility will be specified as CIES (Constant Intertemporal Elasticity of Substitution), so

$$U_t = \frac{c_{1,t}^{1-\frac{1}{\sigma}}}{\left(1-\frac{1}{\sigma}\right)} + \frac{1}{(1+\rho)} \cdot \frac{c_{2,t+1}^{1-\frac{1}{\sigma}}}{\left(1-\frac{1}{\sigma}\right)}, \quad \sigma > 0, \sigma \neq 1. \tag{2}$$

Note that with CIES utility (also know as Constant Relative Risk Aversion or CRRA) marginal utility is[1]

$$U'(c) = c^{-\frac{1}{\sigma}} \tag{3}$$

and the elasticity of marginal utility is constant and equal to $-\frac{1}{\sigma}$:

$$\frac{c}{U'(c)} \cdot U''(c) = \frac{c}{c^{-\frac{1}{\sigma}}} \cdot \left(-\frac{1}{\sigma} c^{-\frac{1}{\sigma}-1}\right) = -\frac{1}{\sigma}. \tag{4}$$

1.2 The Budget

Agents inelastically and successfully supply one unit of labor when young and receive wage income w_t. Since the there are no bequests, the initial assets of the young are zero. The consumption of the young is therefore equal to wage income less saving

$$c_{1,t} = w_t - S_t \tag{5}$$

and the consumption of the old is given by the wealth accumulated from savings when young:

$$\begin{aligned} c_{2,t+1} &= (1 + r_{t+1}) \cdot (w_t - c_{1,t}) \\ &= (1 + r_{t+1}) \cdot S_t \end{aligned} \tag{6}$$

where $S_t \succeq 0$ is the amount of wage income saved when young, and r_{t+1} is the return to savings between periods t and $t+1$. Substituting these consumption values in the two-period Utility program, we obtain:

$$U_t = \frac{(w_t - S_t)^{1-\frac{1}{\sigma}}}{\left(1 - \frac{1}{\sigma}\right)} + \frac{1}{(1+\rho)} \cdot \frac{\left[(1 + r_{t+1}) \cdot S_t\right]^{1-\frac{1}{\sigma}}}{\left(1 - \frac{1}{\sigma}\right)}. \tag{7}$$

[1] At $\sigma = 1$, $U(c)] = \ln c$ by l'Hôpital's rule, although to generate the proof you need to write CIES utility in the first instance as $U(c) = \frac{c^{1-\frac{1}{\sigma}}-1}{\left(1-\frac{1}{\sigma}\right)}$, otherwise the "-1" plays no role, and usually is omitted from $U(c)$, as throughout this lecture.

1.3 The FOC

Wage income w_t and the interest rate r_{t+1} are given, and so the choice ("control") variable determining lifetime utility of consumption is the amount saved out of income when working, S_t. The FOC for lifetime Utility maximization is therefore

$$\frac{\partial U_t}{\partial S_t} = (w_t - S_t)^{-\frac{1}{\sigma}} \cdot (-1) + \frac{1}{(1+\rho)} \cdot [(1+r_{t+1}) \cdot S_t]^{-\frac{1}{\sigma}} \cdot (1+r_{t+1}) = 0, \tag{8}$$

which implies

$$[(1+r_{t+1}) \cdot S_t]^{-\frac{1}{\sigma}} = \frac{(1+\rho)}{(1+r_{t+1})} \cdot (w_t - S_t)^{-\frac{1}{\sigma}}. \tag{9}$$

The left-side term within brackets in (9) is $c_{2,t+1}$, and the second right-side term within parentheses is $c_{1,t}$; after substitution we obtain

$$c_{2,t+1}^{-\frac{1}{\sigma}} = \frac{(1+\rho)}{(1+r_{t+1})} \cdot c_{1,t}^{-\frac{1}{\sigma}}. \tag{10}$$

Note that $U'(c) = c^{-\frac{1}{\sigma}}$ and so eq.(10) implies a relation between marginal utilities at $t+1$ and t that have seen before and will see again:

$$U'[c_{2,t+1}] = \frac{(1+\rho)}{(1+r_{t+1})} \cdot U'[c_{1,t}]. \tag{11}$$

Solving (10) for the growth of lifetime consumption yields

$$\left(\frac{c_{2,t+1}}{c_{1,t}}\right)^{-\frac{1}{\sigma}} = \frac{(1+\rho)}{(1+r_{t+1})} \tag{12}$$

$$\left(\frac{c_{2,t+1}}{c_{1,t}}\right) = \left[\frac{(1+\rho)}{(1+r_{t+1})}\right]^{-\sigma}.$$

Taking logs we obtain

$$\ln\left(\frac{c_{2,t+1}}{c_{1,t}}\right) = \sigma \cdot \ln\left[\frac{(1+r_{t+1})}{(1+\rho)}\right]. \tag{13}$$

1.4 The Saving-Income Relation

From the eq.(9) FOC

$$[(1+r_{t+1})\cdot S_t]^{-\frac{1}{\sigma}} = \frac{(1+\rho)}{(1+r_{t+1})}\cdot(w_t - S_t)^{-\frac{1}{\sigma}}$$

we can find the relation of saving S and wage income w. Showing, as usual, a lot of Mickey Mouse derivation steps, the dependence of saving on income is

$$(1+r_{t+1})\cdot S_t = \frac{(1+\rho)^{-\sigma}}{(1+r_{t+1})^{-\sigma}}\cdot(w_t - S_t)$$

$$S_t\cdot\left[1+\frac{(1+\rho)^{-\sigma}}{(1+r_{t+1})^{1-\sigma}}\right] = \frac{(1+\rho)^{-\sigma}}{(1+r_{t+1})^{1-\sigma}}\cdot w_t$$

$$S_t = \frac{(1+\rho)^{-\sigma}}{\left[\left(1+\frac{(1+\rho)^{-\sigma}}{(1+r_{t+1})^{1-\sigma}}\right)\right]\cdot(1+r_{t+1})^{1-\sigma}}\cdot w_t$$

$$S_t = \frac{1}{\left[(1+\rho)^{\sigma}\cdot(1+r_{t+1})^{1-\sigma}+1\right]}\cdot w_t$$

$$S_t = \beta_t\cdot w_t, \tag{14}$$

where $\beta_t \equiv \frac{1}{\left[1+(1+\rho)^{\sigma}\cdot(1+r_{t+1})^{1-\sigma}\right]} < 1.0$.

In the overlapping generations model the dependence of aggregate saving on aggregate wage income therefore is

$$S_w \equiv \frac{\partial S_t}{\partial w_t} = \beta_t, \qquad 0 < \beta_t < 1. \tag{15}$$

The result could hardly be otherwise. $S_w > 1$ violates the budget constraint – savings cannot exceed earnings. $S_w = 1$ means that nothing is consumed in first period of life, and so the young would starve and never make it to retirement. More illuminating is a boundary case in which we have log utility, $\sigma = 1$, and the future is not discounted, $\rho = 0$. This scenario implies

$$\beta_t \equiv \frac{1}{\left[1+(1+0)^{1}\cdot(1+r_{t+1})^{0}\right]} = \frac{1}{2}, \tag{16}$$

which says that the young put aside half their working life income to finance consumption in old age and, in fact, enjoy higher consumption in retirement than during working life because the income saved would grow by a factor of $(1 + r_{t+1})$.

1.4.1 The Response of Saving to the Interest Rate:

Given $S_t = \beta_t \cdot w_t$ and $\beta_t \equiv \frac{1}{\left[1+(1+\rho)^{\sigma}\cdot(1+r_{t+1})^{1-\sigma}\right]}$ $(eq.14)$, the saving response to an interest rate change, $S_r \equiv \frac{\partial S_t}{\partial r_{t+1}}$, is

$$
\begin{aligned}
S_r &= \frac{\partial S_t}{\partial \beta_t} \cdot \frac{\partial \beta_t}{\partial r_{t+1}} = w_t \cdot \frac{\partial \beta_t}{\partial r_{t+1}} \qquad (17) \\
&= w_t \cdot \left\{ \begin{array}{c} -\left[1+(1+\rho)^{\sigma}\cdot(1+r_{t+1})^{1-\sigma}\right]^{-2} \\ \cdot\left[(1-\sigma)\cdot(1+\rho)^{\sigma}\cdot\left(1+r_{t+1}\right)^{-\sigma}\right] \end{array} \right\} \\
&= w_t \cdot \left\{-\beta_t^2 \cdot \left[(1-\sigma)\cdot(1+\rho)^{\sigma}\cdot\left(1+r_{t+1}\right)^{-\sigma}\right]\right\} \\
&= (\sigma-1)\cdot\left(\frac{(1+\rho)}{\left(1+r_{t+1}\right)}\right)^{\sigma}\cdot\beta_t^2\cdot w_t.
\end{aligned}
$$

Since $B_t S_t = B_t \cdot (\beta_t \cdot w_t) = B_t^2 w_t$, eq.(17)can be written

$$
S_r = (\sigma-1)\cdot\left(\frac{(1+\rho)}{\left(1+r_{t+1}\right)}\right)^{\sigma}\cdot\beta_t\cdot S_t. \qquad (18)
$$

The result has important interpretation. Remember that σ is the intertemporal elasticity of substitution. Hence:

- If $\sigma > 1$, $S_r > 0$: A rise in the interest rate induces optimal agents to reduce current consumption and increase current saving. So $c_{2,t+1}$ becomes marginally more desirable than $c_{1,t}$; future consumption is substituted for current consumption. The "substitution effect" dominates the "income effect" of higher returns to saving.

- If $\sigma < 1$, $S_r < 0$: A rise in the interest rate induces optimal agents to take benefit of more current consumption by reducing current saving. The increased return to saving means that a given level of future consumption, $c_{2,t+1}$, can be achieved with less saving and more consumption during working life. The income effect dominates the substitution effect.

- If $\sigma = 1$, $S_r = 0$: This is the log utility case (the case used in the DGME setup which, as noted therein, is required for general equilibrium). The income and substitution effects exactly offset each other.

2 Firms

We have a closed economy, neoclassical environment and everything is now in discrete time. Production exhibits constant returns to scale (CRS)

$$Q_t = F[K_t, (N_t \cdot A_t)] \tag{19}$$

and may be expressed in "intensive form" as

$$q_t = f(k_t) \tag{20}$$

where

$$\begin{aligned} q &\equiv \frac{Q}{AN}, k \equiv \frac{K}{AN}, f \equiv F(k, 1) \text{ and} \\ f'(k) &> 0, \qquad f''(k) < 0 \\ f'(0) &= \infty, \qquad f'(\infty) = 0. \end{aligned}$$

Technology and the labor force grow at rates

$$\begin{aligned} A_{t+1}/A_t &= (1+g) \\ &\Rightarrow A_{t+1} = (1+g)^{t+1} A_0 \end{aligned} \tag{21}$$

$$N_{t+1}/N_t = (1+n) \qquad (22)$$
$$\Rightarrow \quad N_{t+1} = (1+n)^{t+1} N_0.$$

Profit maximization means firms hire labor up to the point where its marginal product equals the given wage

$$w_t = \frac{\partial Q_t}{\partial N_t} = A_t \cdot \left[f\left(k_t\right) - k_t \cdot f'\left(k_t\right) \right] \qquad (23)$$

and that firms deploy capital up to the point where its user cost equals market interest rate:

$$r_t = f'\left(k_t\right) - \delta. \qquad (24)$$

3 Equilibrium

3.1 The Dynamics of Capital

The capital accumulation equation is

$$K_{t+1} - K_t = w_t N_t + r_t K_t - c_{1,t} N_t - c_{2,t} N_{t-1}.[2] \qquad (25)$$

Now recall that from the household sector we have $c_{1,t} = (w_t - S_t)$ and $c_{2,t} =$

[2] Note that $c_{1,t}N_t$ is the aggregate consumption of young (working) people at time-t. $c_{2,t}N_{t-1}$ is the aggregate consumption of old (retired) people at time-t. In other words, there are N_{t-1} generation-period2 consumers at time-t; they are the people who were workers and savers in the previous generation-period. (Everybody lives to enjoy retirement. Unlike myself, I guess no one smokes.) The sum of the two generations' time-t consumption gives the aggregate, economy-wide level of consumption at time-t. Also note that the market interest rate equals the marginal product of capital less the depreciation rate, $r_t = f'(k_t) - \delta$, and that under CRS production factor payments exhaust output, $w_t N_t + f'(k_t) K_t = Q_t$, where recall $w_t = F'N$ and $f'(k_t) = F'(K)$. Since the economy is closed, aggregate saving, $S_t = Q_t - C_t$, equals aggregate investment. Therefore the capital accumulation equation in the main text indeed corresponds to the more familiar representations:

$(1 + r_t) \cdot S_{t-1}$. So the dynamics of capital can be expressed

$$K_{t+1} - K_t = w_t N_t + r_t K_t - (w_t - S_t) \cdot N_t - (1 + r_t) \cdot S_{t-1} \cdot N_{t-1} \qquad (26)$$

and therefore the stock of capital is

$$\begin{aligned} K_{t+1} &= (1 + r_t) \cdot K_t + w_t N_t - (w_t - S_t) \cdot N_t - (1 + r_t) \cdot S_{t-1} \cdot N_{t-1} \qquad (27) \\ &= (1 + r_t) \cdot (K_t - S_{t-1} \cdot N_{t-1}) + S_t \cdot N_t. \end{aligned}$$

We need an initial condition for the capital stock to get things started at, say, period $t = 1$. The following initial condition scenario is imposed at $t = 1$:

We know from the budget constraint that the second period consumption of people who are old at $t = 1$ equals the return to income saved when they were young, that is, the return to their $t = 0$ saving:

$$c_{2,t=1} \cdot N_{t=0} = (1 + r_{t=1}) \cdot S_{t=0} \cdot N_{t=0}. \qquad (28)$$

Households own the capital stock, and the $t = 1$ aggregate capital stock, $K_{t=1}$, is owned by the $N_{t=0}$ people who are old at $t = 1$, and it is equal to their aggregate savings during period $t = 0$ when they were young and working:

$$K_{t=1} = S_{t=0} \cdot N_{t=0}. \qquad (29)$$

The consumption of those $N_{t=0}$ old people in period $t = 1$ therefore satisfies

$$\begin{aligned} c_{2,t=1} \cdot N_{t=0} &= (1 + r_{t=1}) \cdot S_{t=0} \cdot N_{t=0} \\ &= (1 + r_{t=1}) \cdot K_{t=1}. \end{aligned} \qquad (30)$$

It follows from the capital accumulation equation that at period $t = 2$ the capital

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$$\begin{aligned} K_{t+1} - K_t &= w_t N_t + (f'(k_t) - \delta) K_t - C_t \\ &= Q_t - C_t - \delta K_t \\ &= I_t - \delta K_t. \end{aligned}$$

stock is

$$
\begin{aligned}
K_{t=2} &= (1+r_{t=1}) \cdot (K_{t=1} - S_{t=0} \cdot N_{t=0}) + S_{t=1} \cdot N_{t=1} \\
&= (1+r_{t=1}) \cdot (S_{t=0} \cdot N_{t=0} - S_{t=0} \cdot N_{t=0}) + S_{t=1} \cdot N_{t=1} \\
&= S_{t=1} \cdot N_{t=1}.
\end{aligned} \tag{31}
$$

Hence

$$
K_{t+1} = S_t N_t, \qquad t \succeq 2. \tag{32}
$$

This result implies that capital stock equals the aggregate savings of the young. It follows from generational "selfishness": The old generation owns the capital stock, and the old make no bequests to their descendants. Consequently, as they exit the world to the great beyond, they sell the capital stock to the next generation of retirees. The capital stock therefore must be purchased with the savings of the young. The dynamic replicates itself from generation to generation, yielding the equation above.

Capital per effective worker in the overlapping generations setting is

$$
\begin{aligned}
k_{t+1} &\equiv \frac{K_{t+1}}{A_{t+1}N_{t+1}} = \frac{S_t \cdot N_t}{A_{t+1}N_{t+1}} \\
&= S_t \cdot \left(\frac{N_{t+1}}{N_t}\right)^{-1} \cdot \frac{1}{(1+g) \cdot A_t} \\
&= \frac{S_t}{(1+n)} \cdot \frac{1}{(1+g) \cdot A_t},
\end{aligned} \tag{33}
$$

which follows from our assumptions about the dynamics of labor force growth and technological change: $A_{t+1} = (1+g) \cdot A_t$ and $N_{t+1} = (1+n) \cdot N_t$.

Recall that given CIES utility and neoclassical production

$$
S_t = \beta_t \cdot w_t,
$$

$$
\beta_t \equiv \frac{1}{\left[1 + (1+\rho)^{\sigma} \cdot (1+r_{t+1})^{1-\sigma}\right]}
$$

$$w_t = \frac{\partial Q_t}{\partial N_t} = A_t \cdot [f(k_t) - k_t \cdot f'(k_t)]$$
$$r_t = f'(k_t) - \delta.$$

Hence capital per effective worker can be written

$$\begin{aligned} k_{t+1} &= \beta_t \cdot w_t \cdot \frac{1}{(1+n)\cdot(1+g)\cdot A_t} \qquad (34) \\ &= \frac{A_t \cdot [f(k_t) - k_t \cdot f'(k_t)]}{\left[1 + (1+\rho)^{\sigma} \cdot (1 + f'(k_{t+1}) - \delta)^{1-\sigma}\right]} \cdot \frac{1}{(1+n)\cdot(1+g)\cdot A_t} \\ &= \frac{[f(k_t) - k_t \cdot f'(k_t)]}{\left[1 + (1+\rho)^{\sigma} \cdot (1 + f'(k_{t+1}) - \delta)^{1-\sigma}\right]} \cdot \frac{1}{(1+n)\cdot(1+g)}. \end{aligned}$$

k_{t+1} is therefore a nonlinear difference equation (note that $f'(k_{t+1})$ appears on the right-side), and it can be solved in closed form only in special cases of the production and utility functions.

3.2 Cobb-Douglas Production and Log Utility

We consider the case in which production is Cobb-Douglas and utility is logarithmic ($\sigma = 1$):

$$q = k^{\alpha} \qquad (35)$$

$$u[c] = \ln[c]. \qquad (36)$$

Making these substitutions into the solution above for k_{t+1}, we obtain[3]:

$$k_{t+1} = \frac{\left[k_t^\alpha - k_t \cdot \alpha k_t^{\alpha-1}\right]}{[2+\rho]} \cdot \frac{1}{(1+n)\cdot(1+g)} \tag{37}$$
$$= \frac{(1-\alpha)\, k_t^\alpha}{(2+\rho)\cdot(1+n)\cdot(1+g)}.$$

3.3 The Steady-State

At steady-state, $k_{t+1} = k_t = k^*$. From the solution above for k_{t+1} with Cobb-Douglas production and logarithmic utility, we find

$$k^* = \frac{(1-\alpha)\, k^{*\alpha}}{(2+\rho)\cdot(1+n)\cdot(1+g)} \tag{38}$$
$$= \left[\frac{(1-\alpha)}{(2+\rho)\cdot(1+n)\cdot(1+g)}\right]^{\frac{1}{(1-\alpha)}}.$$

Since $q = k^\alpha$, steady-state output per effective worker is

$$q^* = \left[\frac{(1-\alpha)}{(2+\rho)\cdot(1+n)\cdot(1+g)}\right]^{\frac{\alpha}{(1-\alpha)}}. \tag{39}$$

[3]Note that since $\widetilde{k}_{t+1} = A_{t+1} \cdot k_{t+1}$ and $A_{t+1} = (1+g)\, A_t$, the corresponding equation for the per worker capital stock is just

$$\widetilde{k}_{t+1} = A_{t+1} \cdot \frac{(1-\alpha)\cdot\left(A_t^{-1}\cdot\widetilde{k}_t\right)^\alpha}{(2+\rho)\cdot(1+n)\cdot(1+g)}$$
$$= (1+g)\, A_t \cdot A_t^{-\alpha} \frac{(1-\alpha)\cdot\left(\widetilde{k}_t\right)^\alpha}{(2+\rho)\cdot(1+n)\cdot(1+g)}$$
$$= A_t^{1-\alpha} \frac{(1-\alpha)\cdot\left(\widetilde{k}_t\right)^\alpha}{(2+\rho)\cdot(1+n)}.$$

The corresponding capital per worker and output per worker conditional steady-states, $\tilde{k}_t^* | (k = k^*)$ and $\tilde{q}_t^* | (q = q^*)$, are of course just $A_t \cdot k^*$ and $A_t \cdot q^*$. Bear in mind that the growth rates n and g pertain to rates over half a lifetime in the 2-period OLG model and, consequently, they have correspondingly larger magnitudes than in continuous time models or discrete time modes with conventional periodicity. (See ahead)

[*Graph of capital dynamics*]

Convergence Speed when k is near k^*:

In the Cobb Douglas-logarithmic utility specification of the OLG model it is not hard to learn something about the speed of convergence. The difference equation for k_{t+1} in eq.37 was

$$k_{t+1} = f(k_t) = \theta k_t^\alpha$$

where θ denotes $\frac{(1-\alpha)}{(2+\rho)\cdot(1+n)\cdot(1+g)}$. Taking a first-order Taylor approximation of the function for k_{t+1} at k_t in the vicinity of k^* we have

$$k_{t+1} \simeq \theta k^{\alpha *} + \alpha \theta k^{*(\alpha-1)} \cdot (k_t - k^*) \tag{40}$$

The steady-state result k^*in eq.38 implies that $k^{*(1-\alpha)} = \theta$, and $k^{*(\alpha-1)} = \theta^{-1}$. Hence the approximation above can be written

$$\begin{aligned} k_{t+1} &\simeq k^{*(1-\alpha)} \cdot k^{\alpha *} + \alpha\, \theta \cdot \theta^{-1} \cdot (k_t - k^*) \\ &\simeq k^* + \alpha \cdot (k_t - k^*)\,. \end{aligned} \tag{41}$$

We also can easily derive results for the speed of convergence from some initial condition, just as we did in the analogous exercise with the Solow-Swan model. Eq.41 implies

$$k_{t+1} \cdot (1 - \alpha L) \simeq (1-\alpha) \cdot k^*. \tag{42}$$

Multiplying the equation through by $(1 + \alpha L + \alpha^2 L^2 + .. + \alpha^t L^t)$ we obtain[4]

$$
\begin{aligned}
k_{t+1} - \alpha^{t+1} k_0 &\simeq (1-\alpha) \cdot \frac{(1-\alpha^{t+1})}{(1-\alpha)} k^* \qquad (43) \\
&\simeq \alpha^{t+1} k_0 + \left(1 - \alpha^{t+1}\right) k^*
\end{aligned}
$$

and therefore

$$
k_{t+1} - k^* \simeq \alpha^{t+1} (k_0 - k^*) . \qquad (44)
$$

From overlapping generation to overlapping generation the capital stock approaches steady-state geometrically, with convergence rate that rises as the share of capital income, α, gets smaller.[5] Remember, however, that a "period" is a generation. (See below)

4 Dynamic Inefficiency and the Golden Rule

Remember that the exogenous saving of the Solow-Swan model admits inefficient over-saving ("dynamic inefficiency"). By contrast, over-saving can never occur in the endogenous saving model of Ramsey-Cass-Koopmans with a finite number of infinitely-lived, optimizing households. In the OLG case, saving is also endogenous *but* households have a finite ("2 period") horizon, whereas the economy goes on forever. Consequently, it is possible that on the balanced growth path the capital stock may exceed the golden rule level (a coordination problem that a central social planner could help rectify, although the behavioral record of central investment planning does not, to say the least, inspire confidence).

Recall that the *golden rule saving rate* (producing the highest sustainable level of consumption) generates a capital stock satisfying

$$
f'(k^*) = (\delta + g + n) , \qquad (45)
$$

[4] Cf the lecture notes on 'Lag Algebra and First-Order Difference Equations'.

[5] In fashion analogous to what I did in the lecture on convergence in the Solow-Swan model, one could obtain results in logs, which are often better suited to empirical work, by using the close approximations $\frac{(k_{t+1}-k_t)}{kt} \simeq (\ln k_{t+1} - \ln k_t)$ and $\frac{(k_t - k^*)}{kt} \simeq (\ln k_t - \ln k^*)$. All the dynamics therefore pass through to log variables. The corresponding solution(s) for $\ln q$ would be the same as for $\ln k$ because $\ln q = \alpha \ln k$ and so $(\ln k_t - \ln k^*) = \frac{1}{\alpha} (\ln q_t - \ln q^*)$.

which in the Cobb-Douglas case is

$$\alpha k^{*(\alpha-1)} = (\delta + g + n) \, . \tag{46}$$

So the golden rule Cobb-Douglas capital stock is[6]

$$k^{*}gold = \left[\frac{\alpha}{(\delta + g + n)}\right]^{\frac{1}{(1-\alpha)}} . \tag{47}$$

The OLG steady-state capital stock (eq. 38) is

$$k^{*}OLG = \left[\frac{(1-\alpha)}{(2+\rho)\cdot(1+n)\cdot(1+g)}\right]^{\frac{1}{(1-\alpha)}} .$$

We have dynamic inefficiency ("over-saving") if

$$f'\left(k^{*}OLG\right) < f'\left(k^{*}gold\right), \tag{48}$$

implying that dynamic inefficiency is present when

$$k^{*}OLG \;\; > \;\; k^{*}gold \tag{49}$$

$$\left[\frac{(1-\alpha)}{(2+\rho)\cdot(1+n)\cdot(1+g)}\right] \;\; > \;\; \left[\frac{\alpha}{(\delta+g+n)}\right] .$$

One way to sort out the matter is to plug in some reasonable parameter values for ρ, n, g and δ, remembering that we must express parameters in magnitudes suited to time in generations in order to compare the OLG steady-state capital stock with the golden rule steady-state capital stock. At annual values of $\rho = 0.02$, $n = 0.01$, $g = 0.02$ and $\delta, = 0.05$ the proper values would be approximately 0.81, 0.35, 0.81 and 0.785, respectively,for generations of 30 years.[7]

[6]Recall from the Solow-Swan lecture that the steady-state capital stock for a general neoclassical production function is $k^{*} = \left[\frac{s}{(\delta+g+n)}\right]^{\frac{1}{(1-\alpha)}}$. Under Cobb-Douglas, the golden rule then requires that $s = \alpha$, the share of capital in total output.

[7]For the geometric growth rates and discount rates we want to equate $(1 + x\_Annual)^{30}$ to $(1 + x\_Generation)$, where x_Annual denotes typical annual values for ρ, n and g, and 30 is

Inserting these values in the inequality condition above implies that $k^*OLG > k^*gold$ if

$$(1-\alpha)\cdot 0.15 > \alpha \cdot 0.51, \tag{50}$$

that is, if

$$\alpha < 0.22. \tag{51}$$

Since α is the share of capital in total output, even under a narrow conception of capital (just physical capital) this condition is unlikely to be satisfied. We may conclude that inefficient over-saving is not plausible in the OLG framework with Cobb-Douglas production and CIES utility.

One can also approach the issue empirically and in a more general context, and the same conclusion appears to follow. The golden rule requires $f'(k^*) = (\delta + g + n)$, or $[f'(k^*) - \delta] = (g+n)$, where $(g+n)$ is the growth rate of aggregate output at steady-state, and $f'(k^*) - \delta$ is the equilibrium real market rate of interest – the rate at which firms can borrow in the capital markets (and not the 'risk free' rate at which, say, the US government can borrow). Hence if

$$[f'(k^*) - \delta] = r^* > (g+n)$$

the implication is that saving is below the golden rule level and, therefore, outside

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taken to be the number of periods in a generation. So :

$$x\_Generation = (1 + x\_Annual^{30}) - 1.$$

For the generation-long rate of depreciation, note that

$$K_{t+1} - K_t = I_t - \delta K_t$$

implies

$$\begin{aligned} K_{t+30} &= (1-\delta)^{30} K_t + \sum_{j=0}^{29} (1-\delta)^j I_{t+29-j} \\ K_{t+30} - K_t &= \left[(1-\delta)^{30} - 1\right] K_t + \sum_{j=0}^{29} (1-\delta)^j I_{t+29-j}. \end{aligned}$$

So

$$\delta\_generation = -\left[(1-\delta)^{30} - 1\right].$$

the dynamically inefficient region. In mature economies, output growth rates over the cycle average around 2-3 percent per annum, which casual empiricism suggests is significantly less that the real cost of capital to the typical firm. Romer, chapter 2, provides additional discussion of this issue.

Finally, if the standard OLG model is amended to allow for bequests ("altruism") and the bequest motive is reasonably strong, the model delivers results essentially the same as Ramsey-Cass-Koopmans. See Barro and Sala-i-Martin, chapter 3.8, for a demonstration.

Lecture References

Acemoglu, Modern Economic Growth, 2009, chapter 9.

Barro and Sala-i-Martin, Economic Growth, 2004, chapter 3.8.

Blanchard and Fisher, Lectures in Macroeconomics, 1989, chapter 3.

Romer, Advanced Macroeconomics, 2006, chapter 2.